Non-Linear Stability in the Photogravitational Elliptic Restricted Three Body Problem with Poynting-Robertson Drag

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Abstract. We examined the non-linear stability of triangular equilibrium points in the photogravitational elliptic restricted three body problem with Poynting-Robertson drag. In this problem, the bigger primary is taken as radiating and smaller primary is assumed to be an oblate spheroid. We performed normalization of Hamiltonian of our problem. Using Whittaker (1965) method we have found that the second order part H2 of the Hamiltonian is transformed into the normal form. We have found normalized Hamiltonian up to fourth order. To find the condition of non-linear stability, we have used KAM theorem. We have found three critical mass ratios. We came to conclusion that triangular equilibrium points are stable in the non-linear sense except at three critical mass ratios at which KAM theorem fails.

Keywords: Non-linear stability, triangular equilibrium points, photogravitational, ERTBP, oblateness, P-R drag.

1   Introduction

We have studied non-linear stability of triangular equilibrium points in the photogravitational elliptic restricted three body problem with Poynting-Robertson drag. The elliptic restricted three body problem (ERTBP) is a generalization of the classical problem. The eccentricity of the orbits plays a significant role. In ERTBP, the primaries move in an elliptical orbit in the plane around their common centre of mass. The orbits of most of the celestial bodies are elliptical rather than circular. Thus ERTBP gives more accurate results than circular restricted three body problem (CRTBP).

The primaries in classical CRTBP are strictly spherical in shape but in actual situations, we find that several heavenly bodies such as Saturn and Jupiter are sufficiently oblate. The oblateness of primaries affects the motion of the infinitesimal mass. This motivated many investigators to study CRTBP with oblateness of the primaries, such as Bhatnagar and Chawala (1983), Elipe and Ferrer (1985), Ishwar (1997) and many others. Due to the vital role of oblateness of primaries, we supposed smaller primary as an oblate spheroid in this problem.

The photogravitational effect arises from the classical problem when at least one of the interacting bodies exerts radiation pressure, for example, Sun-Planet system or binary star system. The photogravitational RTBP under different aspects was studied by Radzievskii (1950), Chernikov (1970), Schuerman (1980), Singh and Ishwar (1999) etc. Ishwar (1997) examined non-linear stability in the generalized restricted three body problem when infinitesimal mass is taken as an oblate spheroid. Subba Rao and Sharma (1997) studied effect of oblateness on the non-linear stability of L4 in the RTBP Sahoo and Ishwar(2000) examined stability of collinear equilibrium points in the generalized photogravitational ERTBP. Kumar and Ishwar (2009) investigated the solutions of generalized photogravitational ERTBP. A.Narayan and C.R.Kumar (2011) studied the effect of photogravitational and oblateness on the triangular equilibrium points in the ERTBP. Singh and Umar (2013) located and examined the stability of the out of plane equilibrium points in ERTBP with radiating and oblate primaries. A.Narayan et al. (2015) studied the trajectories of the infinitesimal mass around the triangular equilibrium points in the elliptical restricted three bodies' problem under oblate and radiating primaries for the binary system.

The Poynting-Robertson (P-R) drag named after John Henry Poynting and Howard Peray Robertson, is a process by which solar radiation causes dust grains in a solar system to slowly spiral inward.

Poynting (1903) considered the effect of the absorption and subsequent re-emission of sun light by small isolated particles in the solar system. His work was later modified by Robertson (1937) who used precise relativistic treatments of the first order in the ratio of the velocity of the particle to that of light. Murray (1994) studied the effect of drag forces on the location and stability of five Lagrangian points in the planar CRTBP. The linear stability of triangular equilibrium points in the generalized photo-gravitational restricted three body problems with P-R drag was studied by Ishwar and Kushvah (2006).They found that triangular equilibrium points are unstable due to P-R drag. Kushvah et al. (2007) studied the effect of P-R drag on non linear stability of generalized photogravitational restricted three body problem. They used KAM theorem and found that triangular equilibrium points are stable in non-linear sense except for three critical mass ratios. Mishra and Ishwar (2016) studied linear stability of photogravitational elliptic restricted three body problem with Poynting-Robertson drag.

Mishra and Ishwar (2015) studied first order normalization of Hamiltonian in the photogravitational ERTBP with P-R drag. They found the values of first and second order components are affected by radiation pressure, oblateness and P-R drag. Hence,we aim to study non-linear stability of triangular equilibrium points in photogravitational ERTBP with P-R drag.

2   Equations of Motion

We consider two bodies (primaries) of masses m_1 and m_2 with $m_1 > m_2$ moving in a plane around their common center of mass in elliptic orbit and a third body (infinitesimal mass) of mass m moving in a plane of motion of the primaries. Equations of motion of our problem in rotating and pulsating co-ordinate system are given by Sahoo and Ishwar (2000):

$$x'' - 2y' = \frac{\partial \Omega}{\partial x} - \frac{W_1 N_1}{n^2 r_1^2 \sqrt{1 - e^2}} = U_x \tag{1}$$

$$y'' + 2x' = \frac{\partial \Omega}{\partial y} - \frac{W_1 N_2}{n^2 r_1^2 \sqrt{1 - e^2}} = U_y \tag{2}$$

where the force function

$$U = \frac{1}{\sqrt{1 - e^2}} \left[\frac{x^2 + y^2}{2} + \frac{1}{n^2} \left\{ \frac{(1 - \mu) q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2 r_2^3} + W_1 \left(\frac{(x + \mu) x' + y y'}{2 r_1^2} - n \arctan \left(\frac{y}{x + \mu} \right) \right) \right\} \right]$$

$$\Omega = \frac{1}{\sqrt{1 - e^2}} \left[\frac{x^2 + y^2}{2} + \frac{1}{n^2} \left\{ \frac{(1 - \mu) q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2 r_2^3} \right\} \right]$$

$$N_1 = \frac{(x + \mu) N}{r_1^2} + x' - ny, \quad N_2 = \frac{yN}{r_1^2} + y' - n(x + \mu), \quad N = (x + \mu) x' + y y'$$

$$W_1 = \frac{(1 - \mu)(1 - q_1)}{c_d}$$

Here, dash(') represents differenciation with respect to eccentric anomaly (E).The mean motion of our problem is given by

$$n^2 = \frac{1}{a} \left(1 + \frac{3e^2}{2} + \frac{3A_2}{2} \right)$$

$$r_i = (x + x_i)^2 + y^2 \quad i = 1, 2 \tag{3}$$

$$x_1 = -\mu, x_2 = 1 - \mu, \mu = \frac{m_2}{m_1 + m_2}$$

Here, m_1, m_2 are the masses of the bigger and smaller primaries. $(x_1, 0, 0)$ and $(x_2, 0, 0)$ are the coordinate of m_1 and m_2 respectively. q_1 is mass reduction factor and W_1 is P-R drag due to bigger

primary m_1. $A_2 = \dfrac{r_e^2 - r_p^2}{5r^2}$ is oblateness coefficient due to smaller primary m_2, where r_e, r_p, r represent equatorial radii, polar radii and radius vector respectively. $r_i\,(i = 1,2)$ are the distances of the infinitesimal mass from m_1 and m_2 respectively. Semi-major axis and eccentricity of orbit are denoted by a and e respectively. c_d is dimensionless velocity of light $(c_d = 299792458)$

3   Triangular Equilibrium Points

Using perturbation method, we have found location of triangular equilibrium points. For triangular equilibrium points $U_x = 0$, $U_y = 0$, $y \neq 0$ and $z = 0$ then we have

$$
\begin{aligned}
x = &\frac{1}{2} - \mu + \frac{1}{2}\left[\left(aq_1\right)^{2/3}\left(1 - A_2 - e^2\right) - a^{2/3}(1 - A_2 - e^2 + A_2 a^{-2/3}\right] \\
&+ \frac{W_1 a^{1/2}}{3y_0\left(1 - \mu\right)\mu}\left[\left(1 + \frac{A_2}{4} - \frac{3e^2}{4}\right)\frac{\mu}{2} - \left(a^{2/3} + \left(aq_1\right)^{2/3}\right)\left(1 + \frac{A_2}{4} - \frac{7e^2}{4}\right)\frac{\mu}{2}\right. \\
&\left. - \left(1 - \mu\right)a^{2/3}\left(1 + \frac{A_2}{4} - 2A_2 a^{-2/3} - \frac{7e^2}{4}\right)\right]
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
y = &\pm\left\{\left(aq_1\right)^{2/3}\left(1 - A_2 - e^2\right) - \frac{1}{4}\left[1 + 2\left(\left(aq_1\right)^{2/3} - a^{2/3}\right)\left(1 - e^2\right) + \left(\left(aq_1\right)^{2/3} - a^{2/3}\right)^2\left(1 - 2e^2\right)\right.\right. \\
&\left. - 2A_2\left(1 + \left(aq_1\right)^{2/3} - a^{2/3}\right)^2\right] + \frac{W_1 a^{1/2}}{3y_0\left(1 - \mu\right)\mu}\left[\left(1 + \frac{A_2}{4} - \frac{3e^2}{4}\right)\frac{\mu}{2} - \mu\left(aq_1\right)^{2/3}\left(1 + \frac{A_2}{4} - \frac{7e^2}{4}\right)\right. \\
&\left. + \left(\left(aq_1\right)^{4/3} - a^{4/3}\right)\left(1 + \frac{A_2}{4} - \frac{11e^2}{4}\right)\frac{\mu}{2} + \left(1 - \mu\right)a^{2/3}\left(1 + \frac{A_2}{4} - 2A_2 a^{-2/3} - \frac{7e^2}{4}\right)\right. \\
&\left. + \left(1 - \mu\right)a^{2/3}\left(\left(aq_1\right)^{2/3} - a^{2/3}\right)\left(1 + \frac{A_2}{4} - 2A_2 a^{-2/3} - \frac{11e^2}{4}\right)\right. \\
&\left. - \frac{\mu}{2}A_2\left(1 + \left(aq_1\right)^{2/3} - a^{2/3}\right)\left(\left(aq_1\right)^{2/3} + a^{2/3} - 1\right) - \left(1 - \mu\right)a^{2/3}\left(1 + \left(aq_1\right)^{2/3} - a^{2/3}\right)A_2\right]\right\}^{1/2}
\end{aligned}
\tag{5}
$$

where $y_0 = \pm\left[\delta^2\left(1 - e^2\right) - \dfrac{1}{4}\left\{1 + 2\left(\delta^2 - a^{2/3}\right)\left(1 - e^2\right)\right\}\right]$, $\delta = \left(aq_1\right)^{1/3}$.

4   First Order Normalization

The Lagrangian function of our problem is written as

$$
\begin{aligned}
L = &\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{\dot{v}^2}{2}\left(x^2 + y^2\right) + \dot{v}\left(x\dot{y} - \dot{x}y\right) + \frac{\left(1 - \mu\right)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \\
&+ W_1\left(\frac{\left(x + \mu\right)\dot{x} + y\dot{y} + z\dot{z}}{2r_1^2} - n\arctan\left(\frac{y}{x + \mu}\right)\right)
\end{aligned}
\tag{6}
$$

where v is true anomaly and the Hamiltonian $H = -L + p_x\dot{x} + p_y\dot{y}$, p_x, p_y are the momenta coordinates given by $p_x = \dfrac{\partial L}{\partial \dot{x}} = \dot{x} - \dot{v}y + \dfrac{W_1\left(x + \mu\right)}{2r_1^2}$, $p_y = \dfrac{\partial L}{\partial \dot{y}} = \dot{y} - \dot{v}x + \dfrac{W_1 y}{2r_1^2}$. For simplicity, we suppose $q_1 = 1 - \varepsilon$ with $\varepsilon \ll 1$. Then triangular equilibrium points take the form

$$x = \frac{1}{2} - \mu - \frac{a^{2/3}\varepsilon}{3} + \frac{A_2 a^{2/3}\varepsilon}{3} + \frac{a^{2/3}\varepsilon e^2}{3} - \frac{A_2}{2}$$
$$+ \frac{2\sqrt{3}W_1 a^{1/2}}{9\mu(1-\mu)}\left[\frac{\mu}{6}\left(1 - \frac{3e^2}{4} + \frac{A_2}{4}\right) + \left(1 - \frac{7e^2}{4} + \frac{A_2}{4}\right)\left(\frac{\mu\varepsilon a^{2/3}}{36} - \frac{a^{2/3}}{3}\right)\right.$$
$$- \frac{a^{2/3}}{3}\left(1 - \frac{11e^2}{4} - \frac{3A_2}{4}\right) + \left(1 - \frac{7e^2}{4} - \frac{3A_2}{4}\right)\left(\frac{\mu\varepsilon}{6} - \frac{a^{2/3}\mu}{4}\right)$$
$$\left. + 2A_2(1-\mu)\left(\frac{1}{3} - \frac{a^{2/3}}{2} - \frac{\varepsilon}{3} + \frac{a^{2/3}\varepsilon}{6}\right)\right] \tag{7}$$

$$y = \frac{\sqrt{3}}{2}\left[1 + \frac{a^{2/3}}{2}\left(1 - e^2\right)\left(1 - \frac{\varepsilon}{3}\right) - \frac{A_2}{2}\left(1 + a^{2/3}\right) + \frac{4}{3}\left(A_2 a^{2/3}\varepsilon - 1\right) - \frac{A_2}{2}\right.$$
$$+ \frac{4W_1 a^{1/2}}{9\sqrt{3}\mu(1-\mu)}\left\{\frac{\mu}{6}\left(1 - \frac{3e^2}{4} + \frac{A_2}{4}\right) - \frac{\mu a^{2/3}}{3}\left(1 - \frac{7e^2}{4} + \frac{A_2}{4}\right) + \frac{2\varepsilon a^{2/3}}{9}\left(1 - \frac{7e^2}{4} + \frac{A_2}{4}\right)\right.$$
$$+ (1-\mu)\frac{a^{2/3}}{3}\left(1 - \frac{7e^2}{4} + \frac{A_2}{4} - 2A_2 a^{-2/3}\right) + \frac{\mu A_2}{6}\left(1 + a^{2/3} - \frac{2\varepsilon a^{2/3}}{3} + \frac{2\varepsilon}{3}\right)$$
$$- \frac{A_2 a^{2/3}(1-\mu)}{3} - \frac{\mu a^{2/3}}{4}\left(1 - \frac{7e^2}{4} - \frac{3A_2}{4}\right) - \frac{\mu A_2 a^{2/3}}{4}\left(1 + \frac{2\varepsilon}{3}\right) + \frac{\mu\varepsilon}{6}\left(1 - \frac{7e^2}{4} - \frac{3A_2}{4}\right) \tag{8}$$
$$- \frac{\mu\varepsilon a^{2/3}}{3}\left(1 - \frac{11e^2}{4} - \frac{3A_2}{4}\right) + \frac{\varepsilon a^{2/3}(1-\mu)}{3}\left(1 - \frac{7e^2}{4} + \frac{A_2}{4} - 2A_2 a^{-2/3}\right)$$
$$+ \frac{\mu\varepsilon A_2}{6}\left(1 - e^2 - A_2\right)\left(1 + a^{2/3} - \frac{2\varepsilon a^{2/3}}{3} + \frac{2\varepsilon}{3}\right) - \frac{A_2\varepsilon a^{2/3}}{3}\left(1-\mu\right)\left(1 - e^2 - A_2\right)$$
$$\left. + \frac{\mu\varepsilon a^{2/3}}{12}\left(1 - \frac{7e^2}{4} + \frac{A_2}{4}\right) + \frac{\mu A_2\varepsilon a^{2/3}}{12}\right\}\right]$$

We shift the origin to L_4 (triangular equilibrium point).For that, we change $x \to x_* + x$ and $y \to y_* + y$. We suppose $a_* = x_* + \mu$ and $b_* = y_*$, so that

$$a_* = \frac{1}{2} - \frac{a^{2/3}\varepsilon}{3} + \frac{A_2 a^{2/3}\varepsilon}{3} + \frac{a^{2/3}\varepsilon e^2}{3}$$
$$- \frac{A_2}{2} + \frac{2\sqrt{3}W_1 a^{1/2}}{9\mu(1-\mu)}\left[\frac{\mu}{6}\left(1 - \frac{3e^2}{4} + \frac{A_2}{4}\right) + \left(1 - \frac{7e^2}{4} + \frac{A_2}{4}\right)\left(\frac{\mu\varepsilon a^{2/3}}{36} - \frac{a^{2/3}}{3}\right)\right.$$
$$- \frac{a^{2/3}}{3}\left(1 - \frac{11e^2}{4} - \frac{3A_2}{4}\right) + \left(1 - \frac{7e^2}{4} - \frac{3A_2}{4}\right)\left(\frac{\mu\varepsilon}{6} - \frac{a^{2/3}\mu}{4}\right) \tag{9}$$
$$\left. + 2A_2(1-\mu)\left(\frac{1}{3} - \frac{a^{2/3}}{2} - \frac{\varepsilon}{3} + \frac{a^{2/3}\varepsilon}{6}\right)\right]$$

$$b_* = \frac{\sqrt{3}}{2}\left[1 + \frac{a^{2/3}}{2}\left(1-e^2\right)\left(1-\frac{\varepsilon}{3}\right) - \frac{A_2}{2}\left(1+a^{2/3}\right) + \frac{4}{3}\left(A_2 a^{2/3}\varepsilon - 1\right) - \frac{A_2}{2} \right.$$

$$+ \frac{4W_1 a^{1/2}}{9\sqrt{3}\mu\left(1-\mu\right)}\left\{\frac{\mu}{6}\left(1-\frac{3e^2}{4}+\frac{A_2}{4}\right) - \frac{\mu a^{2/3}}{3}\left(1-\frac{7e^2}{4}+\frac{A_2}{4}\right) + \frac{2\varepsilon a^{2/3}}{9}\left(1-\frac{7e^2}{4}+\frac{A_2}{4}\right)\right.$$

$$+ \left(1-\mu\right)\frac{a^{2/3}}{3}\left(1-\frac{7e^2}{4}+\frac{A_2}{4}-2A_2 a^{-2/3}\right) + \frac{\mu A_2}{6}\left(1+a^{2/3}-\frac{2\varepsilon a^{2/3}}{3}+\frac{2\varepsilon}{3}\right)$$

$$- \frac{A_2 a^{2/3}\left(1-\mu\right)}{3} - \frac{\mu a^{2/3}}{4}\left(1-\frac{7e^2}{4}-\frac{3A_2}{4}\right) - \frac{\mu A_2 a^{2/3}}{4}\left(1+\frac{2\varepsilon}{3}\right) + \frac{\mu\varepsilon}{6}\left(1-\frac{7e^2}{4}-\frac{3A_2}{4}\right) \quad (10)$$

$$- \frac{\mu\varepsilon a^{2/3}}{3}\left(1-\frac{11e^2}{4}-\frac{3A_2}{4}\right) + \frac{\varepsilon a^{2/3}\left(1-\mu\right)}{3}\left(1-\frac{7e^2}{4}+\frac{A_2}{4}-2A_2 a^{-2/3}\right)$$

$$+ \frac{\mu\varepsilon A_2}{6}\left(1-e^2-A_2\right)\left(1+a^{2/3}-\frac{2\varepsilon a^{2/3}}{3}+\frac{2\varepsilon}{3}\right) - \frac{A_2\varepsilon a^{2/3}}{3}\left(1-\mu\right)\left(1-e^2-A_2\right)$$

$$\left.\left.+ \frac{\mu\varepsilon a^{2/3}}{12}\left(1-\frac{7e^2}{4}+\frac{A_2}{4}\right) + \frac{\mu A_2\varepsilon a^{2/3}}{12}\right\}\right]$$

Expanding L in power series of x and y, we get

$$L = L_0 + L_1 + L_2 + L_3 \qquad (11)$$

$$H = H_0 + H_1 + H_2 + H_3 = -L + p_x \dot{x} + p_y \dot{y} \qquad (12)$$

where L_0, L_1, L_2, L_3 are constant, first order, second order, third order terms respectively.

$$L_2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \dot{v}\left(x\dot{y} - \dot{x}y\right) + \frac{\dot{v}^2}{2}\left(x^2 + y^2\right) + Ex^2 + Gxy + Fy^2 \qquad (13)$$

$$L_3 = -\frac{1}{3!}\left\{ x^3 T_1 + 3x^2 y T_2 + 3xy^2 T_3 + y^3 T_4 \right\} \qquad (14)$$

$$L_4 = -\frac{1}{4!}\left\{ x^4 N_1 + 4x^3 y N_2 + 6x^2 y^2 N_3 + 4xy^3 T_4 \right\} \qquad (15)$$

where E, F, G are same as in Mishra and Ishwar (2015), T_i, N_i are given in Appendix1 ($i = 1,2,3,4$).

The second order part H_2 of the corresponding Hamiltonian takes the form

$$H_2 = \frac{p_x^2 + p_y^2}{2} + \dot{v}\left(yp_x - xp_y\right) + Ex^2 + Gxy + Fy^2 \qquad (16)$$

Now, we perform transformation from the phase space (x, y, p_x, p_y) into the phase space ($\varphi_1, \varphi_2, I_1, I_2$) with the help of Whittaker (1965) method. We suppose set of linear equations of variables x and y

$$-\lambda p_x = \frac{\partial H_2}{\partial x} \qquad \lambda_x = \frac{\partial H_2}{\partial p_x}$$

$$-\lambda p_y = \frac{\partial H_2}{\partial y} \qquad \lambda_y = \frac{\partial H_2}{\partial p_y}$$

that is $AX = 0$

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} \qquad A = \begin{bmatrix} 2E & G & \lambda & -n \\ G & 2F & n & y \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{bmatrix}$$

Clearly $\left|A\right|=0$ implies that the characterstic equation corresponding to Hamiltonian is given by

$$\lambda^4 + 2\left(E+F+n^2\right)\lambda^2 + 4EF - G^2 + n^4 - 2n^2\left(E+F\right) = 0 \tag{17}$$

Discriminant is given by

$$D = 4(E+F+n^2)^2 - 44EF - G^2 + n^4 - 2n^2(E+F) \tag{18}$$

Stability is assured only when $D>0$. When $D>0$ the roots $\pm i\omega_1$ and $\pm i\omega_2$ are related to each other as

$$
\begin{aligned}
\omega_1^2 + \omega_2^2 = &\frac{2}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}\right) + \frac{a^{2/3}}{8}\left[-5 - 3a^{2/3} + \varepsilon - 5\varepsilon a^{2/3} - 17A_2 - 7A_2 a^{2/3} + \frac{127A_2\varepsilon a^{2/3}}{6}\right. \\
&+\frac{5}{4}\mu e^2 a^{2/3} + \frac{e^2\varepsilon}{4}\left(1-a^{2/3}\right) - \frac{W_1 a^{2/3}}{3\sqrt{3}}\left(1+\frac{3e^2}{4}+\frac{3A_2}{4}\right)\left\{-10 - 5\gamma - \frac{140\varepsilon}{9} - 12\gamma\varepsilon\right. \\
&+\frac{a^{2/3}}{3}\left(-33 - 6\gamma - \frac{67\varepsilon}{3} - 30\gamma\varepsilon\right) + \frac{a^{4/3}}{3}\left(66 - 3\gamma - \frac{5\varepsilon}{3} - 10\gamma\varepsilon\right)\right\} \\
&+\gamma\left\{-\varepsilon a^{2/3} - 10A_2 - 3\varepsilon + 16A_2\varepsilon + 22A_2 a^{2/3} + \frac{11A_2\varepsilon a^{2/3}}{18}\right. \\
&+\frac{W_1 a^{2/3}}{3\sqrt{3}}\left(1+\frac{3e^2}{4}+\frac{3A_2}{4}\right)\left\{3 - 3\gamma + \frac{514\varepsilon}{9} - \frac{20\gamma\varepsilon}{3} + \frac{a^{2/3}}{3}\left(-21 - \frac{\gamma}{2} + \frac{168\varepsilon}{3} - \frac{94\gamma\varepsilon}{3}\right)\right. \\
&+\left.\left.\left.\frac{a^{4/3}}{3}\left(\frac{39}{2} + \frac{10\varepsilon}{3} - \frac{11\gamma}{2} - \frac{32\gamma\varepsilon}{3}\right)\right\}\right\}\right]
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\omega_1^2\omega_2^2 = &\frac{19a^{2/3}}{16} - \frac{1}{2} - \frac{5\gamma^2}{8} - \frac{17\gamma^2 a^{2/3}}{16} + \frac{9\varepsilon}{8} + \frac{9\gamma\varepsilon a^{2/3}}{8} - \frac{\gamma^2\varepsilon}{4} - \frac{\gamma^2\varepsilon a^{2/3}}{8} + \frac{97\gamma A_2}{16} + \frac{5\gamma A_2 a^{2/3}}{4} \\
&+\frac{1}{a^2}\left(1+3A_2+3e^2\right) - 3\gamma A_2 a^{2/3} - \frac{177A_2\varepsilon}{32} - 2A_2\varepsilon a^{2/3} + \frac{2485\gamma\varepsilon A_2 a^{2/3}}{192} - \frac{5000\gamma\varepsilon A_2}{192} + \frac{25e^2\gamma\varepsilon}{16} \\
&+\frac{7e^2\gamma\varepsilon a^{2/3}}{16} - \frac{34e^2\gamma^2\varepsilon}{16} + \frac{W_1 a^{2/3}}{3\sqrt{3}}\left(1+\frac{3e^2}{4}+\frac{3A_2}{4}\right)\left\{\frac{15}{16} - \frac{216\gamma}{16} - \frac{27\gamma^2}{8} - \frac{2225\varepsilon}{192} - \frac{5\varepsilon}{96} - \frac{3260\gamma\varepsilon}{96}\right. \\
&+\frac{900\gamma^2\varepsilon}{16} + \frac{a^{2/3}}{3}\left(\frac{45}{8} - \frac{315\gamma}{16} - \frac{60\gamma^2}{8} - \frac{525\varepsilon}{192} - \frac{2700\gamma\varepsilon}{96} + \frac{99\gamma^2\varepsilon}{16}\right) \\
&+\left.\frac{a^{4/3}}{3}\left(\frac{90}{8} - \frac{432\varepsilon}{192} - \frac{522\gamma}{16} - \frac{129\gamma^2}{8} - \frac{2583\gamma\varepsilon}{96} + \frac{99\gamma^2\varepsilon}{16}\right)\right\}
\end{aligned}
\tag{20}
$$

Following the method for reducing H_2 to the normal form, as in Whittaker (1965), we use the transformation $X = JT$ where

$$
X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} \qquad T = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix}
$$

and $J = \left[J_{ij}\right]_{1\le i,j\le 4}$ is a matrix of order four.

$$P_i = \left(2I_i\omega_i\right)^{1/2}\cos\varphi_i \qquad Q_i = \left(\frac{2I_i}{\omega_i}\right)^{1/2}\sin\varphi_i$$

We follow the approach of Breakwell and Pringle (1966) and choose $J_{11} = J_{12} = 0$, H_2 takes the form

$$H_2 = \frac{1}{2}\left(P_1^2 - P_2^2 + \omega_1^2 Q_1^2 - \omega_2^2 Q_2^2\right)$$ (21)

and

$$J = \begin{bmatrix} 0 & 0 & \dfrac{-M_1}{\omega_1 M_{1*}} & \dfrac{iM_2}{\omega_2 M_{2*}} \\[2mm] \dfrac{-2n\omega_1}{M_1 M_{1*}} & \dfrac{-2n\omega_2}{M_2 M_{2*}} & \dfrac{G}{\omega_1 M_1 M_{1*}} & \dfrac{iG}{\omega_2 M_2 M_{2*}} \\[2mm] \dfrac{-\omega_1\left(M_1^2 - 2n^2\right)}{M_1 M_{1*}} & \dfrac{i\omega_2\left(M_2^2 - 2n^2\right)}{M_2 M_{2*}} & \dfrac{-nG}{\omega_1 M_1 M_{1*}} & \dfrac{niG}{\omega_2 M_2 M_{2*}} \\[2mm] \dfrac{-\omega_1 G}{M_1 M_{1*}} & \dfrac{iG\omega_2}{\omega_2 M_2 M_{2*}} & \dfrac{ni\left(\omega_1^2 - M_1^2\right)}{\omega_1 M_1 M_{1*}} & \dfrac{ni\left(\omega_2^2 - M_2^2\right)}{\omega_2 M_2 M_{2*}} \end{bmatrix}$$

where $M_j = \left(\omega_j^2 - 2F + n^2\right)^{1/2}$ and $M_{j*} = \sqrt{2}\left(\omega_j^2 - E - 2n^2\right)^{1/2}$, $j = 1,2$

Applying a contact transformation form Q_1, Q_2, P_1, P_2 to $Q_1^{''}, Q_2^{''}, P_1^{''}, P_2^{''}$ defined by Whittaker (1965)

$P_j^{''} = \dfrac{\partial W}{\partial Q_j}$, $Q_j^{''} = \dfrac{\partial W}{\partial P_j}$, $j = 1,2$ and $W = \sum_{j=1}^{2}\left[Q_j^{''}\sin^{-1}\left(\dfrac{P_j}{\sqrt{2\omega_j Q_j^{''}}}\right) + \dfrac{P_j}{2\omega_j}\sqrt{2\omega_j Q_j^{''} - P_j^2}\right]$ that is $Q_j = \sqrt{\dfrac{2Q_j^{''}}{\omega_j}}\cos P_j^{''}$,

$P_j = \sqrt{2\omega_j Q_j^{''}}\sin P_j^{''}$, $j = 1,2$.

The transformation changes the second order part of the Hamiltonian into normal form

$$H_2 = \omega_1 I_1 - \omega_2 I_2$$ (22)

5   Second Order Coefficients in the Frequencies

Using Henrard's method (1990), we observe that the third order components $B_3^{1,0}$ and $B_3^{0,1}$ in the coordinates x, y and the second order polynomials f_2 and g_2 in the frequencies $\dot{\varphi}_1$ and $\dot{\varphi}_2$ satisfy the partial differential equations

$$\Delta_1 \Delta_2 B_3^{1,0} = \phi_3 - 2f_2 P - 2g_2 Q$$ (23)

$$\Delta_1 \Delta_2 B_3^{0,1} = \psi_3 - 2f_2 U - 2g_2 V$$ (24)

where

$$\phi_3 = \left[D^2 + 2F - n^2\right]X_3 + \left[2nD - G\right]Y_3$$ (25)

$$\psi_3 = -\left[2nD + G\right]X_3 + \left[D^2 + 2nE - n^2\right]Y_3$$ (26)

$$P = \left[D^2 + 2F - n^2\right]\left[\omega_1 \frac{\partial^2 B_1^{1,0}}{\partial^2 \varphi_1^2} - n\frac{\partial B_1^{1,0}}{\partial \varphi_1}\right] + \left[2nD - G\right]\left[\omega_1 \frac{\partial^2 B_1^{1,0}}{\partial^2 \varphi_1^2} + n\frac{\partial B_1^{0,1}}{\partial \varphi_1}\right]$$ (27)

$$Q = -\left[D^2 + 2F - n^2\right]\left[\omega_2 \frac{\partial^2 B_1^{1,0}}{\partial^2 \varphi_1^2} - n\frac{\partial B_1^{0,1}}{\partial \varphi_1}\right] - \left[2nD - G\right]\left[\omega_2 \frac{\partial^2 B_1^{0,1}}{\partial^2 \varphi_2^2} + n\frac{\partial B_1^{1,0}}{\partial \varphi_1}\right]$$ (28)

$$U = \left[D^2 + 2E - n^2\right]\left[\omega_1 \frac{\partial^2 B_1^{0,1}}{\partial^2 \varphi_1^2} + n\frac{\partial B_1^{1,0}}{\partial \varphi_1}\right] - \left[2nD + G\right]\left[\omega_1 \frac{\partial^2 B_1^{1,0}}{\partial \varphi_1^2} - n\frac{\partial B_1^{0,1}}{\partial \varphi_1}\right]$$ (29)

$$V = -\left[D^2 + 2E - n^2\right]\left[\omega_2 \frac{\partial^2 B_1^{0,1}}{\partial \varphi_2^2} - n\frac{\partial B_1^{1,0}}{\partial \varphi_2}\right] - \left[2nD + G\right]\left[\omega_2 \frac{\partial^2 B_1^{1,0}}{\partial \varphi_2^2} - n\frac{\partial B_1^{0,1}}{\partial \varphi_2}\right]$$ (30)

$$X_3 = \frac{\partial L_3}{\partial x} + \frac{\partial L_4}{\partial x}, Y_3 = \frac{\partial L_3}{\partial y} + \frac{\partial L_4}{\partial y}$$ (31)

E, F are same as in Mishra & Ishwar(2015). X_3, Y_3 are the homogeneous components of order three

obtained by putting $x = B_1^{1,0} + B_2^{1,0}, y = B_1^{0,1} + B_2^{0,1}$ in $\dfrac{\partial L_3}{\partial x} + \dfrac{\partial L_4}{\partial x}$ and $\dfrac{\partial L_3}{\partial y} + \dfrac{\partial L_4}{\partial y}$.

The coefficients $B_3^{1,0}$ and $B_3^{0,1}$ are not required to be found out. We find the coefficients of $\cos\varphi_1, \sin\varphi_1, \cos\varphi_2$ and $\sin\varphi_2$ in the RHS of equations (25) and (26). They are the critical terms .We eliminate these terms by choosing properly the coefficients in the polynomials

$$f_2 = f_{2,0} I_1 + f_{0,2} I_2 \text{ and } g_2 = g_{2,0} I_1 + g_{0,2} I_2$$

where

$$f_{2,0} = \frac{1}{2} \frac{\text{Coefficients of } \cos\varphi_1 \text{ in } \phi_3}{\text{Coefficients of } \cos\varphi_2 \text{ in } P} = A \tag{32}$$

$$f_{0,2} = g_{2,0} = \frac{1}{2} \frac{\text{Coefficients of } \cos\varphi_2 \text{ in } \phi_3}{\text{Coefficients of } \cos\varphi_2 \text{ in } Q} = B \tag{33}$$

$$g_{0,2} = \frac{1}{2} \frac{\text{Coefficients of } \cos\varphi_2 \text{ in } \psi_3}{\text{Coefficients of } \cos\varphi_2 \text{ in } Q} = A \tag{34}$$

with

$$A = A_{1,1} + \left(A_{1,2} + A_{1,3}\gamma\right)\varepsilon + \left(A_{1,4} + A_{1,5}\gamma\right)A_2 + \left(A_{1,6} + A_{1,7}\gamma\right)W_1 + \left(A_{1,8} + A_{1,9}\gamma\right)W_1 e^2 \tag{35}$$

$$B = B_{1,1} + \left(B_{1,2} + B_{1,3}\gamma\right)\varepsilon + \left(B_{1,4} + B_{1,5}\gamma\right)A_2 + \left(B_{1,6} + B_{1,7}\gamma\right)W_1 + \left(B_{1,8} + B_{1,9}\gamma\right)W_1 e^2 \tag{36}$$

$$C = C_{1,1} + \left(C_{1,2} + C_{1,3}\gamma\right)\varepsilon + \left(C_{1,4} + C_{1,5}\gamma\right)A_2 + \left(C_{1,6} + C_{1,7}\gamma\right)W_1 + \left(C_{1,8} + C_{1,9}\gamma\right)W_1 e^2 \tag{37}$$

where $A_{1,i}$, $B_{1,i}$ and $C_{1,i}$ are as in Appendix 2.

6 Stability

For this, we used KAM [Kolmogrov(1954),Arnold(1961),Moser(1962)] theorem. This theorem states that

   i) If the basic frequencies ω_1 and ω_2 for the linear dynamical system satisfy the inequalities

   $k_1\omega_1 + k_2\omega_2 \neq 0$ for all pairs of integers(k_1, k_2) such that $|k_1| + |k_2| \leq 4$

   ii) $D = \det\left(d_{ij}\right) \neq 0$, where

$$d_{ij} = \frac{\partial^2 H}{\partial I_i \partial I_i}, I_i = 0, I_j = 0, \left(i, j = 1, 2\right)$$

$$d_{i,3} = d_{3,i} = \frac{\partial H}{\partial I_i}, \left(i = 1, 2\right), d_{3,3} = 0 \text{ and}$$

$$H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}\left(A I_1^2 + 2B I_1 I_2 + C I_2^2\right) + \ldots\ldots$$

The Normalized Hamiltonian with I_1 and I_2 as the action momenta coordinates, then the equilibrium point is stable for all orders.

Now

$$k_1\omega_1 + k_2\omega_2 = 0 \Leftrightarrow \frac{\omega_1}{\omega_2} = -\frac{k_2}{k_1} \tag{38}$$

Since $0 < \omega_2 < \dfrac{1}{\sqrt{2}} < \omega_1 < 1$, so $\dfrac{\omega_1}{\omega_2} > 1$.

Equation (38) is true if k_1 and k_2 are of opposite signs and $\dfrac{k_2}{k_1} > 1$. Thus, two cases arise.

$$\omega_1 = 2\omega_2 \quad \text{and} \quad \omega_1 = 3\omega_2 \tag{39}$$

Case 1

$$\omega_1 = 2\omega_2 \tag{40}$$

With the help of equations (40) and (20), we get

$$\mu^2 \left\{ \frac{-18}{4} - 5\frac{\varepsilon}{2} - 98\frac{A_2}{4} - \frac{a^{2/3}}{2}\left(\frac{9}{2} - 2\varepsilon + 19\frac{A_2}{2} \right) - \frac{W_1 a^{1/2}}{3\sqrt{3}}\left(\frac{97}{5} + \frac{255a^{2/3}}{15} + \frac{39a^{4/3}}{5} + \frac{111e^2 a^{2/3}}{5} \right) \right\}$$
$$+ \mu \left\{ \frac{9}{2} - \frac{49\varepsilon}{50} + \frac{1057A_2}{100} + \frac{a^{2/3}}{2}\left(\frac{9}{2} - \frac{9\varepsilon}{50} + \frac{1964A_2}{50} \right) + \frac{W_1 a^{1/2}}{3\sqrt{3}}\left(\frac{1435}{40} + \frac{2378a^{2/3}}{40} + \frac{478a^{4/3}}{40} + \frac{517e^2 a^{2/3}}{40} \right) \right\} \tag{41}$$
$$+ \frac{391\varepsilon}{200} - \frac{36A_2}{25} - \frac{a^{2/3}}{3}\left(\frac{12}{25} - \frac{48\varepsilon}{200} - \frac{24A_2}{8} \right) - \frac{W_1 a^{1/2}}{3\sqrt{3}}\left(\frac{25341}{200} + \frac{29200a^{2/3}}{200} + \frac{17432a^{4/3}}{200} + \frac{1581e^2 a^{2/3}}{200} \right) = 0$$

Solving it for μ we get

$$\mu_{c_1} = 0.13456521 + 0.010837376a^{2/3} - 0.2132543172\varepsilon$$
$$- 0.0994380079\varepsilon a^{2/3} - 0.023450124311A_2 - 0.013400433311A_2 a^{2/3} \tag{42}$$
$$+ 0.62563948959W_1 + 0.37541234521W_1 a^{2/3} + 0.723568230512W_1 e^2$$

Case 2

$$\omega_1 = 3\omega_2 \tag{43}$$

With the help of equations (43) and (20), we get

$$\mu^2 \left\{ \frac{-18}{4} - 5\frac{\varepsilon}{2} - 98\frac{A_2}{4} - \frac{a^{2/3}}{2}\left(\frac{9}{2} - 2\varepsilon + 19\frac{A_2}{2} \right) - \frac{W_1 a^{1/2}}{3\sqrt{3}}\left(\frac{327}{5} + \frac{247a^{2/3}}{5} + \frac{417a^{4/3}}{5} + \frac{185e^2 a^{2/3}}{5} \right) \right\} +$$
$$\mu \left\{ \frac{9}{2} - \frac{107\varepsilon}{100} + \frac{1821A_2}{100} + \frac{a^{2/3}}{2}\left(\frac{9}{2} + \frac{14\varepsilon}{50} + \frac{1158A_2}{50} \right) + \frac{W_1 a^{\frac{1}{2}}}{3\sqrt{3}}\left(\frac{573}{10} + \frac{257a^{2/3}}{10} + \frac{241a^{4/3}}{10} + \frac{117e^2 a^{2/3}}{10} \right) \right\} - \tag{44}$$
$$\frac{1}{20} + \frac{178\varepsilon}{200} - \frac{57A_2}{100} - \frac{a^{2/3}}{3}\left(\frac{3}{25} - \frac{645\varepsilon}{200} + \frac{9A_2}{10} \right) - \frac{W_1 a^{1/2}}{3\sqrt{3}}\left(\frac{5371}{400} + \frac{6378a^{2/3}}{400} + \frac{2582a^{4/3}}{400} + \frac{337e^2 a^{2/3}}{200} \right) = 0$$

Solving it for μ, we have

$$\mu_{c_2} = 0.000357016031 + 0.01315899991a^{2/3} - 0.184132201327\varepsilon$$
$$- 0.113111200132\varepsilon a^{2/3} - 0.0161310111141A_2 - 0.0032520104042A_2 a^{2/3} \tag{45}$$
$$+ 1.0031411111014W_1 + 0.0045412230164W_1 a^{2/3} + 0.6231532207323W_1 e^2$$

Normalized Hamiltonian up to fourth order is

$$H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}\left(A I_1^2 + 2B I_1 I_2 + C I_2^2 \right) + \ldots\ldots \tag{46}$$

Calculating the determinant D occurring in condition (ii) of KAM theorem, we get

$$D = -\left(A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2 \right) \tag{47}$$

Putting the value of A, B and C and suppose $u = \omega_1\omega_2$, we have

$$D = \frac{\left(644u^4 - 541u^2 + 36 \right)}{8\left(4u^2 - 1 \right)\left(25u^2 - 4 \right)} + \left(D_2 + D_3\gamma \right)\varepsilon + \left(D_4 + D_5\gamma \right)A_2 + \left(D_6 + D_7\gamma \right)W_1 + \left(D_8 + D_9\gamma \right)W_1 e^2 \tag{48}$$

$D_i\left(i = 2\ldots9 \right)$ are given as Appendix 3.

Condition (ii) of KAM theorem is violated for the unperturbed problem i.e. for $A_2 = \varepsilon = W_1 = 0$ when $\mu_0 = 0.0109136\ldots\ldots$ When $A_2 \neq 0, W_1 \neq 0, \varepsilon \neq 0$, we take $\mu = \mu_0 + \alpha_1\epsilon + \alpha_2 A_2 + \alpha_3 W_1 + \alpha_4 W_1 e^2$, such

that $D = 0$. We find that condition (ii) of KAM theorem is satisfied, that is $D \neq 0$ if in the interval $0 < \mu < \mu_{c_0}$, the mass ratio does not take the value

$$\mu_{c_3} = \mu_0 + \alpha_1 \varepsilon + \alpha_2 A_2 + \alpha_3 W_1 + \alpha_4 W_1 e^2 \tag{49}$$

where

$$\alpha_1 = -0.00327602011510314 - 0.11721311001111222 a^{2/3} \tag{50}$$

$$\alpha_2 = -0.13100311513414132 - 0.24211500225325132 a^{2/3} \tag{51}$$

$$\alpha_3 = 1.2011303236313412 + 1.7031603511210532 a^{2/3} \tag{52}$$

$$\alpha_4 = 1.003572531276523 + 1.32762500120035 a^{2/3} \tag{53}$$

Hence,

$$
\begin{aligned}
\mu_{c_3} = {} & 0.01091366767720066 - \left(0.00327602011510314 + 0.11721311001111222 a^{2/3}\right)\varepsilon \\
& - \left(0.13100311513414132 + 0.24211500225325132 a^{2/3}\right)A_2 \\
& + \left(1.2011303236313412 + 1.7031603511210532 a^{2/3}\right)W_1 \\
& + \left(1.003572531276523 + 1.32762500120035 a^{2/3}\right)W_1 e^2
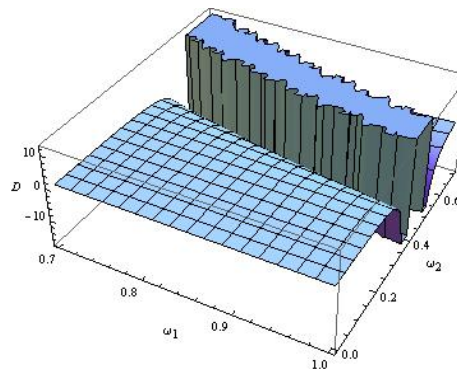\end{aligned}
\tag{54}
$$

Hence in the interval $0 < \mu < \mu_{c_0}$, both the conditions of KAM theorem are satisfied and therefore the triangular points are stable except for three mass ratios μ_{c_i} $(i = 1, 2, 3)$.

7   Analytical discussion

7.1   Observation I

When we consider $e = 0, a = 1, A_2 = 0$ and $q_1 = 1$ then this problem reduced to classical restricted three body problem. We get $x = \frac{1}{2} - \mu$ and $y = \pm \frac{\sqrt{3}}{2}$ from equations (4) and (5), which are coordinate of triangular equilibrium points in classical case. Equations (19) and (20) give $\omega_1^2 + \omega_2^2 = 1$ and $\omega_1^2 \omega_2^2 = \frac{27}{10} - \gamma^2$ $(\gamma = 1 - 2\mu)$. From equations (42), (45) and (54), we obtain that triangular equilibrium points are stable in the range of linear stability except the three mass ratios $\mu_{c1} = 0.24294, \mu_{c_2} = 0.013516, \mu_{c_3} = 0.010914$ and from second condition of KAM theorem, we get $D = \frac{\left(644u^4 - 541u^2 + 36\right)}{8\left(4u^2 - 1\right)\left(25u^2 - 4\right)}$. 3D diagram between $D - \omega_1 - \omega_2$ is shown by Fig.1.
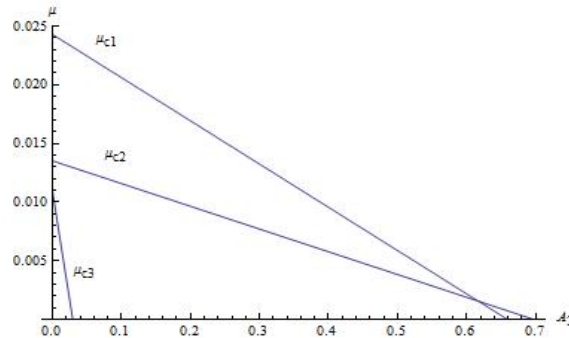

Figure 1. $e = 0, a = 1, A_2 = 0, q_1 = 1$

All these results are exactly similar to Deprit-Bartholome(1967).

7.2   Observation II

When we consider $e = 0, a = 1, A_2 \neq 0, q_1 = 1$, i.e smaller primary is taken as an oblate spheroid and bigger primary is not radiating. Then from equations (42), (45) and (54), we get.

Figure 2. Stability region $\mu_{c_i}\left(i = 1, 2, 3\right) - A_2$, when $e = 0, a = 1, q_1 = 1$, $\mu_{c_1} = 0.024294 - 0.036851A_2$,

$$\mu_{c_2} = 0.013516 - 0.019383A_2, \quad \mu_{c_3} = 0.010914 - 0.373118A_2$$

We have drawn $\mu_{c_i} - A_2$. Fig 2 shows that μ decreases as A_2 increases. These results are similar to those of Markellos (1996).

7.3   Observation III

If we consider $e = 0, a = 1$ then our problem reduced to photogravitational restricted three body problems with P-R drag, i.e primaries moves in circular orbit about their common centre of mass. With this assumption, we get from equation (4) and (5),

$$x = \frac{q_1^{2/3}}{2} - \mu - \frac{nW_1}{3\mu\left(1 - \mu\right)y_0}\left[\frac{\mu q_1^{2/3}}{2}\left(1 - \frac{A_2}{2}\right) + \left(1 - \mu\right)\left(1 - \frac{5A_2}{2}\right)\right] - \frac{A_2 q_1^{2/3}}{2}$$
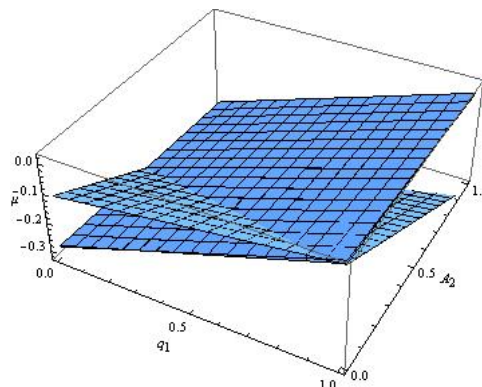
$$y = \pm\left[q_1^{2/3} - \frac{q_1^{4/3}}{4} - \frac{nW_1 q_1^{2/3}}{3\mu\left(1 - \mu\right)y_0}\left\{2\mu - 1 - \frac{\mu q_1^{2/3}}{2}\left(1 - \frac{3A_2}{2}\right) + \frac{7A_2}{2}\left(1 - \mu\right) - \frac{A_2\mu}{2}\right\} - A_2 q_1^{2/3}\left(1 - \frac{q_1^{2/3}}{2}\right)\right]^{1/2}$$

which are location of triangular equilibrium points in circular case. From equations (42), (45) and (54), we have found that the triangular equilibrium points are stable in the range of linear stability except the three mass ratios

$$\mu_{c_1} = 0.24294 - 0.312692\varepsilon - 0.036851A_2 + 1.001052W_1 \tag{55}$$

$$\mu_{c_2} = 0.013516 - 0.29724\varepsilon - 0.019383A_2 + 1.007682W_1 \tag{56}$$

$$\mu_{c_3} = 0.010914 - 0.120489\varepsilon - 0.373118A_2 + 2.904291W_1 \tag{57}$$

Figure 3. Stability region $\mu_{c_i} - q_1 - A_2$

With the help of above three equations (55), (56) and (57) we have drawn $\mu - A_2 - q_1$ 3D diagram. In Fig.3 uppermost plane, middle plane and innermost plane are due to μ_{c_1}, μ_{c_2} and μ_{c_3} All above results exactly similar to Kushvah and Ishwar (2006, 2007).

7.4   Observation IV

When we consider our problem i.e $e \neq 0, a \neq 1, A_2 \neq 0, q_1 \neq 1$ then location of triangular equilibrium points are given by equations (4) and (5).They are affected by radiation, oblateness and eccentricity of orbit. Putting $\varepsilon = 1 - q_1$, $W_1 = \dfrac{(1-\mu)\varepsilon}{c_d}$, $\mu = \mu_{c_i}$ $(i = 1,2,3)$ in equations (42,45,54),we have

$$
\begin{aligned}
\mu_{c_1} = {} & 0.013456521 + 0.010837376a^{2/3} - 0.023450124311A_2 - 0.013400433311a^{2/3}A_2 \\
& - 0.2132543172(1-q_1) - 0.0994380079a^{2/3}(1-q_1) \\
& + \frac{0.62563948959\left(0.986543479 - 0.010837376a^{2/3}\right)(1-q_1)}{c_d} \\
& + \frac{0.37541234521\left(0.986543479 - 0.010837376a^{2/3}\right)a^{2/3}(1-q_1)}{c_d} \\
& + \frac{0.723568230512\left(0.986543479 - 0.010837376a^{2/3}\right)e^2(1-q_1)}{c_d}
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
\mu_{c_2} = {} & 0.000357016031 + 0.01315899991a^{2/3} - 0.01613101111141A_2 - 0.0032520104042a^{2/3}A_2 \\
& - 0.184132201327(1-q_1) - 0.113111200132a^{2/3}(1-q_1) \\
& + \frac{1.0031411111014\left(0.999642983969 - 0.01315899991a^{2/3}\right)(1-q_1)}{c_d} \\
& + \frac{0.\left(0.999642983969 - 0.01315899991a^{2/3}\right)a^{2/3}(1-q_1)}{c_d} \\
& + \frac{0.6231532207323\left(0.999642983969 - 0.01315899991a^{2/3}\right)e^2(1-q_1)}{c_d}
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
\mu_{c_3} = {} & 0.01091366767720066 - 0.1310031151341413A_2 - 0.24211500225325133a^{2/3}A_2 \\
& - 0.00327602011510314(1-q_1) - 0.11721311001111222a^{2/3}(1-q_1) \\
& + \frac{1.1880215864422201(1-q_1)}{c_d} + \frac{1.6845726250479336a^{2/3}(1-q_1)}{c_d} \\
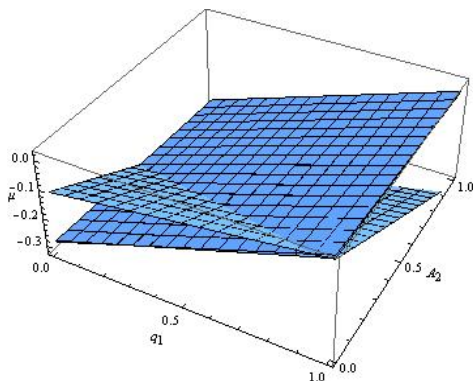& + \frac{0.9926198741802038e^2(1-q_1)}{c_d}
\end{aligned} \tag{60}
$$

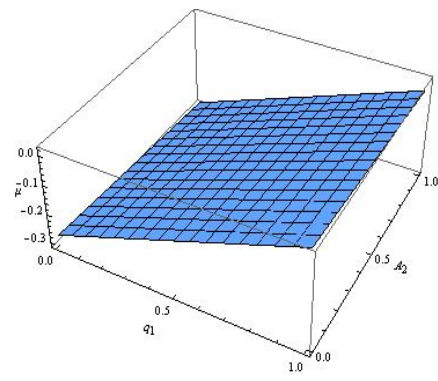**Figure 4.** Stability region $\quad \mu_{c_1} - q_1 - A_2$

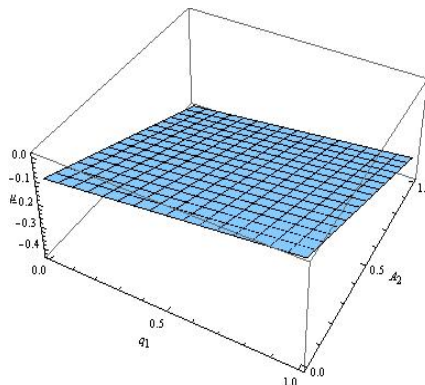**Figure 5.** Stability region $\quad \mu_{c_2} - q_1 - A_2$

Figure 6. Stability region $\quad \mu_{c_3} - q_1 - A_2$

Using above equations with, we have drawn $\mu_{c_i} - A_2 - q_1$, 3D diagram Plane due to μ_{c_1} is shown by Fig.4. Planes due to μ_{c_2} and μ_{c_3} are given by Fig 5 and Fig 6 respectively. From these figures, we conclude that stability region is reduced due to eccentricity of orbit. We have found that triangular equilibrium points are stable in the range of linear stability except three mass ratios () at which KAM theorem fails.

8   Conclusion

We conclude that triangular equilibrium points are stable in the non- linear sense for all ratios in the range of linear stability except for three mass ratios. These critical mass ratios are given by equations (42),(45) and (54).We also conclude that stability region is reduced due to P-R drag, oblateness and eccentricity of orbit.

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Appendix 1

Coefficients of L_3 and L_4 .

$$T_1 = \frac{a^{2/3}}{16}\left[\begin{array}{l} \dfrac{7\varepsilon}{3} + \dfrac{9\varepsilon a^{2/3}}{3} + A_2 + 5A_2 a^{2/3} - \dfrac{1805 A_2 \varepsilon}{18} + \dfrac{1805 A_2 \varepsilon a^{2/3}}{18} \\[2mm] + \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right) \\[3mm] \left\{\begin{array}{l}\dfrac{21}{3} + \dfrac{5\gamma}{2} + \dfrac{134\varepsilon}{9} + \dfrac{103\gamma\varepsilon}{9} + \dfrac{a^{2/3}}{3}\left(\dfrac{28}{3} + \dfrac{24\gamma}{2} + \dfrac{615\varepsilon}{9} + \dfrac{615\gamma\varepsilon}{9}\right)\\[2mm] + \dfrac{a^{4/3}}{3}\left(\dfrac{33}{3} + \dfrac{12\gamma}{2} + \dfrac{648\varepsilon}{9} + \dfrac{204\gamma\varepsilon}{9}\right)\end{array}\right\} \\[5mm] +\gamma\left\{\begin{array}{l} 8 + 6a^{2/3} + \dfrac{2\varepsilon}{3} + \dfrac{2\varepsilon a^{2/3}}{3} + 25A_2 a^{2/3} - \dfrac{3042 A_2 \varepsilon}{18} + \dfrac{1535 A_2 \varepsilon a^{2/3}}{18} - \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right) \\[2mm] \left\{8 + 6a^{2/3} + \dfrac{2\varepsilon}{3} + \dfrac{2\varepsilon a^{2/3}}{3} + 25A_2 a^{2/3} - \dfrac{3042 A_2 \varepsilon}{18} + \dfrac{1535 A_2 \varepsilon a^{2/3}}{18} - \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right)\right\}\end{array}\right\}\end{array}\right]$$

$$T_2 = \frac{3\sqrt{3}a^{2/3}}{16}\left[\begin{array}{l} 10 + 4a^{\frac{2}{3}} - \dfrac{8\varepsilon}{3} - \dfrac{8\varepsilon a^{\frac{2}{3}}}{3} + \dfrac{A_2 a^{\frac{2}{3}}}{3} - \dfrac{367 A_2 \varepsilon a^{\frac{2}{3}}}{18} + \dfrac{W_1 a^{\frac{1}{2}}}{3\sqrt{3}} \\[3mm] \left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right) \\[3mm] \left\{\begin{array}{l}\dfrac{93}{6} + \dfrac{93\gamma}{6} + \dfrac{305\varepsilon}{9} + \dfrac{75\gamma\varepsilon}{9} + \dfrac{a^{2/3}}{3} + \dfrac{93\gamma}{6} + \dfrac{305\varepsilon}{9} + \dfrac{75\gamma\varepsilon}{9} + \dfrac{a^{2/3}}{3}\\[2mm] \left(\dfrac{21}{6} + \dfrac{21\gamma}{6} + \dfrac{993\varepsilon}{9} + \dfrac{75\gamma\varepsilon}{9}\right) + \dfrac{a^{4/3}}{3}\left(\dfrac{45}{6} + \dfrac{45\gamma}{6} + \dfrac{969\varepsilon}{9} + 12\gamma\varepsilon)\right)\end{array}\right\} \\[5mm] +\left\{\begin{array}{l}\left[\dfrac{37\varepsilon}{3} + 40A_2 - \dfrac{382 A_2 \varepsilon a^{2/3}}{9} - \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right)\right. \\[2mm] \left\{\left[\dfrac{333}{2} + \dfrac{23\gamma}{2} + \dfrac{1125\varepsilon}{9} + \dfrac{149\gamma\varepsilon}{9} + \dfrac{a^{2/3}}{3}\left(\dfrac{321}{2} + \dfrac{75\gamma}{2} + \dfrac{2355\varepsilon}{9} + \dfrac{153\gamma\varepsilon}{9}\right) + \right.\right. \\[2mm] \left.\left. \dfrac{a^{4/3}}{3}\left(\dfrac{213}{2} + \dfrac{15\gamma}{2} + \dfrac{1827\varepsilon}{9} + \dfrac{147\gamma\varepsilon}{9}\right)\right]\right\}\end{array}\right\}\end{array}\right]$$

$$T_3 = \frac{9a^{2/3}}{16}\left[\begin{array}{l} \varepsilon a^{2/3} + \dfrac{5\varepsilon}{3} + \dfrac{155A_2}{6} + \dfrac{48A_2 a^{2/3}}{3} - \dfrac{352A_2\varepsilon}{54} \\[2mm] - \dfrac{273A_2\varepsilon a^{2/3}}{54} - \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right) \\[3mm] \left\{\begin{array}{l} \dfrac{55}{6} + \dfrac{8\gamma}{6} + \dfrac{203\varepsilon}{27} - \dfrac{53\gamma\varepsilon}{27} \\[2mm] + \dfrac{a^{2/3}}{3}\left(\dfrac{135}{6} + \dfrac{18\gamma}{6} + \dfrac{453\varepsilon}{27} - \dfrac{141\gamma\varepsilon}{27}\right) + \dfrac{a^{4/3}}{3}\left(\dfrac{15}{6} + \dfrac{3\gamma}{6} + \dfrac{147\varepsilon}{27} - \dfrac{42\gamma\varepsilon}{27}\right) \end{array}\right\} \\[6mm] +\gamma\left\{\begin{array}{l} 2 - \dfrac{4\varepsilon a^{2/3}}{9} + \dfrac{27A_2}{2} + 14A_2 a^{2/3} - \dfrac{737A_2\varepsilon}{54} - \dfrac{10A_2\varepsilon a^{2/3}}{9} \\[3mm] + \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right)\left\{\begin{array}{l}\dfrac{57}{6} + \dfrac{33\gamma}{6} - \dfrac{441\varepsilon}{27} + \dfrac{39\gamma\varepsilon}{27} + \dfrac{a^{2/3}}{3}\left(\dfrac{291}{6} + \dfrac{21\gamma}{6} - \dfrac{159\varepsilon}{27} + \dfrac{45\gamma\varepsilon}{9}\right) \\[3mm] + \dfrac{a^{4/3}}{3}\left(\dfrac{129}{6} + \dfrac{15\gamma}{6} + \dfrac{33\varepsilon}{27} + \dfrac{12\gamma\varepsilon}{27}\right)\end{array}\right\} \end{array}\right\} \end{array}\right]$$

$$T_4 = -\frac{9\sqrt{3}a^{2/3}}{16}\left[\begin{array}{l} 2a^{2/3} - \dfrac{5\varepsilon}{3} - \varepsilon a^{2/3} + \dfrac{23A_2 a^{2/3}}{3} - 47A_2\varepsilon + 3\varepsilon A_2 a^{2/3} \\[3mm] - \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right) \\[3mm] \left\{\begin{array}{l}\dfrac{17}{6} + \dfrac{17\gamma}{6} + \dfrac{157\varepsilon}{27} + \dfrac{139\gamma\varepsilon}{27} + \dfrac{a^{2/3}}{3}\left(\dfrac{39}{6} - \dfrac{36\gamma}{6} + \dfrac{129\varepsilon}{27} + \dfrac{183\gamma\varepsilon}{27}\right) + \\[3mm] \dfrac{a^{4/3}}{3}\left(\dfrac{21}{6} + \dfrac{18\gamma}{6} + \dfrac{57\varepsilon}{27} + \dfrac{159\gamma\varepsilon}{27}\right)\end{array}\right\} \\[6mm] +r\left\{\begin{array}{l}\left[\dfrac{5\varepsilon}{2} + \dfrac{3\varepsilon a^{2/3}}{2} + 3A_2\varepsilon + \dfrac{7A_2\varepsilon a^{2/3}}{27} + \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right)\right] \\[3mm] \left\{\begin{array}{l}\dfrac{127}{6} + \dfrac{21\gamma}{6} + \dfrac{552\varepsilon}{27} + \dfrac{43\gamma\varepsilon}{27} + \dfrac{a^{2/3}}{3}\left(\dfrac{219}{6} + \dfrac{93\gamma}{6} + \dfrac{1344\varepsilon}{27} + \dfrac{171\gamma\varepsilon}{9}\right) + \\[3mm] \dfrac{a^{4/3}}{3}\left(\dfrac{123}{6} - \dfrac{21\gamma}{6} + \dfrac{1674\varepsilon}{27} + \dfrac{78\gamma\varepsilon}{27}\right)\end{array}\right\}\end{array}\right\} \end{array}\right]$$

$$N_1 = -\frac{a^{2/3}}{2}\left[\begin{array}{l} \dfrac{67}{8} + \dfrac{50a^{2/3}}{8} + \dfrac{201\varepsilon}{16} + \dfrac{84}{16}\varepsilon a^{2/3} - \dfrac{4707}{32}A_2 a^{2/3} - \dfrac{2571}{16}A_2\varepsilon - \dfrac{2636}{16}\varepsilon A_2 a^{2/3} \\[3mm] + \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right) \\[3mm] \left\{\dfrac{441}{32} + \dfrac{5273\varepsilon}{9} + \dfrac{a^{2/3}}{3}\left(\dfrac{711}{32} + \dfrac{8157\varepsilon}{9}\right) + \dfrac{a^{4/3}}{3}\left(\dfrac{711}{32} + \dfrac{8157\varepsilon}{9}\right)\right\} \\[5mm] +\gamma\left\{\begin{array}{l}\left[-\dfrac{115\varepsilon a^{2/3}}{16} + \dfrac{1097A_2}{32} + \dfrac{700A_2 a^{2/3}}{32} - \dfrac{5035A_2\varepsilon}{16} - \dfrac{82A_2\varepsilon a^{2/3}}{16} - \dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1 + \dfrac{3A_2}{4} + \dfrac{3e^2}{4}\right)\right] \\[3mm] \left\{\dfrac{221}{4} + \dfrac{4051\varepsilon}{32} + \dfrac{a^{2/3}}{3}\left(\dfrac{537}{4} + \dfrac{4605\varepsilon}{32}\right) + \dfrac{a^{4/3}}{3}\left(\dfrac{285}{4} + \dfrac{4395\varepsilon}{32}\right)\right\}\end{array}\right\} \end{array}\right]$$

$$N_2 = -\frac{a^{2/3}}{2}\left[\begin{array}{l}\dfrac{17}{8}+\dfrac{74a^{2/3}}{8}+\dfrac{211\varepsilon}{16}+\dfrac{841}{16}\varepsilon a^{2/3}-\dfrac{4507}{32}A_2 a^{2/3}-\dfrac{1571}{16}A_2\varepsilon-\dfrac{2336}{16}\varepsilon A_2 a^{2/3}\\[2mm]+\dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1+\dfrac{3A_2}{4}+\dfrac{3e^2}{4}\right)\\[2mm]\left\{\dfrac{41}{32}+\dfrac{5273\varepsilon}{9}+\dfrac{a^{2/3}}{3}\left(\dfrac{71}{32}+\dfrac{87\varepsilon}{9}\right)+\dfrac{a^{4/3}}{3}\left(\dfrac{11}{32}+\dfrac{57\varepsilon}{9}\right)\right\}\\[3mm]+\gamma\left\{\begin{array}{l}-\dfrac{11\varepsilon a^{2/3}}{16}+\dfrac{197A_2}{32}+\dfrac{704A_2 a^{2/3}}{32}-\dfrac{35A_2\varepsilon}{16}-\dfrac{82A_2\varepsilon a^{2/3}}{16}-\dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1+\dfrac{3A_2}{4}+\dfrac{3e^2}{4}\right)\\[2mm]\left\{\dfrac{21}{4}+\dfrac{451\varepsilon}{32}+\dfrac{a^{2/3}}{3}\left(\dfrac{57}{4}+\dfrac{465\varepsilon}{32}\right)+\dfrac{a^{4/3}}{3}\left(\dfrac{85}{4}+\dfrac{395\varepsilon}{32}\right)\right\}\end{array}\right\}\end{array}\right]$$

$$N_3 = -\frac{a^{2/3}}{2}\left[\begin{array}{l}\dfrac{67}{4}+\dfrac{50a^{2/3}}{8}+\dfrac{21\varepsilon}{6}+\dfrac{84}{16}\varepsilon a^{2/3}-\dfrac{47}{2}A_2 a^{2/3}-\dfrac{71}{16}A_2\varepsilon-\dfrac{2636}{16}\varepsilon A_2 a^{2/3}\\[2mm]+\dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1+\dfrac{3A_2}{4}+\dfrac{3e^2}{4}\right)\\[2mm]\left\{\dfrac{44}{12}+\dfrac{5273\varepsilon}{9}+\dfrac{a^{2/3}}{3}\left(\dfrac{711}{32}+\dfrac{8157\varepsilon}{9}\right)+\dfrac{a^{4/3}}{3}\left(\dfrac{811}{32}+\dfrac{857\varepsilon}{3}\right)\right\}\\[3mm]+\gamma\left\{\begin{array}{l}-\dfrac{115\varepsilon a^{2/3}}{16}+\dfrac{17A_2}{32}+\dfrac{70A_2 a^{2/3}}{12}-\dfrac{5045A_2\varepsilon}{16}-\dfrac{82A_2\varepsilon a^{2/3}}{16}-\\[2mm]\dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1+\dfrac{3A_2}{4}+\dfrac{3e^2}{4}\right)\left\{\dfrac{22}{14}+\dfrac{4051\varepsilon}{32}+\dfrac{a^{2/3}}{3}\left(\dfrac{57}{12}+\dfrac{605\varepsilon}{12}?\right)+\dfrac{a^{4/3}}{3}\left(\dfrac{85}{4}+\dfrac{495\varepsilon}{32}\right)\right\}\end{array}\right\}\end{array}\right]$$

$$N_4 = -\frac{a^{2/3}}{2}\left[\begin{array}{l}\dfrac{97}{8}+\dfrac{501a^{2/3}}{8}+\dfrac{241\varepsilon}{16}+\dfrac{84}{16}\varepsilon a^{2/3}-\dfrac{4747}{32}A_2 a^{2/3}-\dfrac{771}{16}A_2\varepsilon-\dfrac{9836}{16}\varepsilon A_2 a^{2/3}\\[2mm]+\dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1+\dfrac{3A_2}{4}+\dfrac{3e^2}{4}\right)\\[2mm]\left\{\dfrac{551}{32}+\dfrac{5273\varepsilon}{9}+\dfrac{a^{2/3}}{3}\left(\dfrac{61}{32}+\dfrac{57\varepsilon}{9}\right)+\dfrac{a^{4/3}}{3}\left(\dfrac{711}{32}+\dfrac{8127\varepsilon}{9}\right)\right\}\\[3mm]+\gamma\left\{\begin{array}{l}-\dfrac{115\varepsilon a^{2/3}}{12}+\dfrac{1097A_2}{32}+\dfrac{735A_2 a^{2/3}}{32}-\dfrac{5035A_2\varepsilon}{16}-\dfrac{82A_2\varepsilon a^{2/3}}{16}-\\[2mm]\dfrac{W_1 a^{1/2}}{3\sqrt{3}}\left(1+\dfrac{3A_2}{4}+\dfrac{3e^2}{4}\right)\left\{\dfrac{1}{4}+\dfrac{4051\varepsilon}{32}+\dfrac{a^{2/3}}{3}\left(\dfrac{53}{4}+\dfrac{4605\varepsilon}{32}?\right)+\dfrac{a^{4/3}}{3}\left(\dfrac{285}{4}+\dfrac{4395\varepsilon}{32}\right)\right\}\end{array}\right\}\end{array}\right]$$

Appendix 2

Coefficients of $\varepsilon, A_2, W_1, W_1 e^2$ in A, B, C

$$A_{1,1} = \frac{1}{\left(2\omega_1^2-1\right)^2\left(5\omega_1^2-1\right)}\left[\frac{259\omega_1^2}{24}-\frac{7}{8}-\frac{a^{2/3}}{4}-\frac{125\omega_1^4}{18}+\frac{48\omega_1^6}{18}-\frac{40a^{2/3}}{9}-\frac{17\omega_1^6 a^{2/3}}{18}\right]$$

$$A_{1,2} = \frac{1}{\left(-2\omega_1^2+1\right)^2\left(5\omega_1^2-1\right)}\left[\begin{array}{l}\dfrac{1}{36}-\dfrac{25\omega_1^2}{18}+\dfrac{12\omega_1^2 a^{2/3}}{18}+\dfrac{10\omega_1^4}{27}+\dfrac{3\omega_1^4 a^{2/3}}{27}\\[2mm]+\dfrac{105\omega_1^6}{72}+\dfrac{62\omega_1^6 a^{2/3}}{72}+\dfrac{37\omega_1^8 a^{2/3}}{108}+\dfrac{35\omega_1^8}{54}\end{array}\right]$$

$$A_{1,3} = \frac{1}{\left(-2\omega_1^2+1\right)^3\left(5\omega_1^2-1\right)^2}\left[\begin{array}{c}\dfrac{-1}{2}+\dfrac{1+a^{2/3}}{2}-\dfrac{421\omega_1^2}{32}-\dfrac{17\omega_1^4}{2}-\omega_1^4 a^{2/3}\\[2mm]-\dfrac{8141559\omega_1^6}{32}+9\omega_1^8 a^{2/3}+20\omega_1^8\end{array}\right]$$

$$A_{1,4} = \frac{1}{\left(-2\omega_1^2+1\right)^2\left(5\omega_1^2-1\right)}\left[\begin{array}{c}\dfrac{1300}{436}+\dfrac{19a^{2/3}}{436}-\dfrac{12632\omega_1^2}{436}-\dfrac{7\omega_1^2 a^{2/3}}{436}\\[2mm]+\dfrac{14125\omega_1^4}{436}+\dfrac{150\omega_1^4 a^{2/3}}{436}-\dfrac{721\omega_1^6}{218}-\dfrac{39\omega_1^6 a^{2/3}}{109}\end{array}\right]$$

$$A_{1,5} = \frac{1}{\left(-2\omega_1^2+1\right)^3\left(5\omega_1^2-1\right)^2}\left[\dfrac{57a^{2/3}}{52}-\dfrac{525\omega_1^2}{52}-\dfrac{475\omega_1^4}{26}+\dfrac{1559\omega_1^6}{26}+\dfrac{283\omega_1^6 a^{2/3}}{13}\right]$$

$$A_{1,6} = \frac{1}{\left(2\omega_1^2-1\right)}\left[-\dfrac{2705\omega_1^2}{10368\sqrt{3}}-\dfrac{42\omega_1^2 a^{2/3}}{10368\sqrt{3}}-\dfrac{84510\omega_1^2}{331776\sqrt{3}}-\dfrac{9384\omega_1^2 a^{2/3}}{331776\sqrt{3}}\right]$$
$$+\frac{1}{\left(2\omega_1^2-1\right)^2}\left[\left(\dfrac{41}{9216\sqrt{3}}+\dfrac{12875\omega_1^2 a^{2/3}}{82944\sqrt{3}}\right)\left(9+4\omega_1^2\right)\right]$$

$$A_{1,7} = \frac{1}{\left(2\omega_1^2-1\right)}\left[\dfrac{-932}{6144\sqrt{3}}-\dfrac{405a^{2/3}}{6144\sqrt{3}}-\left(\dfrac{227347\omega_1^6}{331776\sqrt{3}}+\dfrac{27543}{82944\sqrt{3}}+\dfrac{9716a^{2/3}}{82944\sqrt{3}}\right)\left(9+4\omega_1^2\right)\right]$$
$$+\frac{1}{\left(2\omega_1^2-1\right)^2}\left[\left(\dfrac{41a^{2/3}}{18432\sqrt{3}}+\dfrac{54\omega_1 a^{2/3}}{10368\sqrt{3}}+\dfrac{725\omega_1}{10368\sqrt{3}}\right)\left(9+4\omega_1^2\right)\right]$$
$$+\frac{625\omega_1^2 a^{2/3}\left(9+4\omega_1^2\right)}{3072\sqrt{3}\left(-1+2\omega_1^2\right)^2\left(4\omega_1^2-\omega_2^2\right)}+\frac{5892\omega_1^2\left(9+4\omega_1^2\right)}{3072\sqrt{3}\left(-1+2\omega_1^2\right)^2\left(4\omega_1^2-\omega_2^2\right)}$$

$$A_{1,8} = \frac{1}{\left(2\omega_1^2-1\right)\left(5\omega_1^2-1\right)}\left[\dfrac{150\omega_1^4 a^{2/3}}{436}-\dfrac{721\omega_1^6}{218}+\dfrac{41\left(9+4\omega_1^2\right)}{9216\sqrt{3}}-\dfrac{42\omega_1^2 a^{2/3}}{10368\sqrt{3}}+\dfrac{75\omega_1^6}{18}\right]$$
$$+\frac{725\omega_1^2 a^{2/3}\left(9+4\omega_1^2\right)}{3072\sqrt{3}\left(-1+2\omega_1^2\right)\left(4\omega_1^2-\omega_2^2\right)}+\frac{5732a^{2/3}\left(9+4\omega_1^2\right)}{3072\sqrt{3}\left(-1+2\omega_1^2\right)\left(4\omega_1^2-\omega_2^2\right)}$$

$$A_{1,9} = \frac{1}{\left(5\omega_1^2-1\right)^2}\left[-\dfrac{124\omega_1^4 a^{2/3}}{18}+\dfrac{13\omega_1^2}{18}+\dfrac{1390}{435}+\dfrac{3150\omega_1^4 a^{2/3}}{436}+\dfrac{35\omega_1^8 a^{2/3}}{108}+\dfrac{937\omega_1^2}{6144\sqrt{3}}\right]$$
$$+\frac{a^{2/3}}{\left(-2\omega_1^2+1\right)^2\left(5\omega_1^2-1\right)^2}\left[\begin{array}{c}\dfrac{2805\omega_1^2}{33177\sqrt{3}}+\dfrac{12632\omega_1^6}{43251\sqrt{3}}+\dfrac{1237+405\omega_1^2}{6144\sqrt{3}}\\[2mm]+\left(\dfrac{39}{18432\sqrt{3}}-\dfrac{7843}{1443\sqrt{3}}+\dfrac{9716}{10368\sqrt{3}}\right)\left(9+4\omega_1^2\right)\end{array}\right]$$

$$B_{1,1} = \frac{1}{\left(1-5\omega_1^2\right)\left(2\omega_1^2-1\right)\left(1-5\omega_2^2\right)\left(1-2\omega_2^2\right)}\left[\dfrac{83\omega_1\omega_2}{6}-\dfrac{20\omega_1\omega_2 a^{2/3}}{3}+5\omega_1^3\omega_2^3+\dfrac{17\omega_1^3\omega_2^3 a^{2/3}}{3}\right]$$

$$B_{1,2} = \frac{1}{\left(1-5\omega_1^2\right)\left(2\omega_1^2-1\right)\left(1-5\omega_2^2\right)\left(1-2\omega_2^2\right)}\left[\dfrac{305\omega_1\omega_2}{8}+\dfrac{\omega_1\omega_2 a^{2/3}}{2}-\dfrac{315\omega_1^3\omega_2^3}{6}-\dfrac{46\omega_1^3\omega_2^3 a^{2/3}}{3}\right]$$
$$+\frac{540\omega_1\omega_2 a^{2/3}}{\left(2\omega_1^2-1\right)\left(9+4\omega_2^2\right)^2}$$

$$B_{1,3} = \frac{1}{\left(2\omega_1^{\ 2}-1\right)\left(9+4\omega_2^{\ 2}\right)^2}\left[725\omega_1\omega_2 + 1075\omega_1\omega_2 a^{2/3}\right]$$

$$+ \frac{825 - 53475\omega_1^{\ 2}\omega_2^{\ 2} + 237081\omega_1^{\ 4}\omega_2^{\ 4} - 27543\omega_1^{\ 6}\omega_2^{\ 6} - 175827\omega_1^{\ 8}\omega_2^{\ 8}}{8\omega_1\omega_2\left(9 - 59\omega_1^{\ 2} + 62\omega_1^{\ 4} + 40\omega_1^{\ 6}\right)\left(9 - 59\omega_2^{\ 2} + 62\omega_2^{\ 4} + 40\omega_2^{\ 6}\right)}$$

$$+ \frac{a^{\frac{2}{3}}\left(9258 - 56059\omega_1^{\ 2}\omega_2^{\ 2} + 163719\omega_1^{\ 4}\omega_2^{\ 4} - 275978\omega_1^{\ 6}\omega_2^{\ 6} - 84975\omega_1^{\ 8}\omega_2^{\ 8}\right)}{8\omega_1\omega_2\left(9 - 59\omega_1^{\ 2} + 62\omega_1^{\ 4} + 40\omega_1^{\ 6}\right)\left(9 - 59\omega_2^{\ 2} + 62\omega_2^{\ 4} + 40\omega_2^{\ 6}\right)}$$

$$B_{1,4} = \frac{1}{\left(1-5\omega_1^{\ 2}\right)\left(-2\omega_1^{\ 2}+1\right)\left(1-5\omega_2^{\ 2}\right)\left(1-2\omega_2^{\ 2}\right)}\left[\frac{178\omega_1\omega_2}{4} + \frac{69\omega_1\omega_2 a^{2/3}}{4} + \frac{9325\omega_1^{\ 3}\omega_2^{\ 3}}{36} - \frac{2508\omega_1^{\ 3}\omega_2^{\ 3}a^{2/3}}{36}\right]$$

$$B_{1,5} = \frac{-75431 + 1045668\omega_1^{\ 2}\omega_2^{\ 2} + 57823\omega_1^{\ 4}\omega_2^{\ 4} + 1143752\omega_1^{\ 6}\omega_2^{\ 6} + 345273\omega_1^{\ 8}\omega_2^{\ 8}}{32\omega_1\omega_2\left(-1+5\omega_1^{\ 2}\right)^2\left(-1+5\omega_2^{\ 2}\right)^2\left(9+14\omega_1^{\ 2}+8\omega_1^{\ 4}\right)\left(-9-14\omega_2^{\ 2}+18\omega_2^{\ 4}\right)}$$

$$+ \frac{a^{2/3}\left(-13780 + 997330\omega_1^{\ 2}\omega_2^{\ 2} + 97075\omega_1^{\ 4}\omega_2^{\ 4} + 1490834\omega_1^{\ 6}\omega_2^{\ 6} + 870531\omega_1^{\ 8}\omega_2^{\ 8}\right)}{32\omega_1\omega_2\left(-1+5\omega_1^{\ 2}\right)^2\left(-1+5\omega_2^{\ 2}\right)^2\left(9+14\omega_1^{\ 2}+8\omega_1^{\ 4}\right)\left(-9-14\omega_2^{\ 2}+18\omega_2^{\ 4}\right)}$$

$$B_{1,6} = \frac{\sqrt{3}\left(9 + 192\omega_1\omega_2 + \omega_2^{\ 2}\right)\left(3575 - 1976a^{2/3}\right)}{512\omega_1\omega_2\left(-1+2\omega_1^{\ 2}\right)\left(9+4\omega_2^{\ 2}\right)^2}$$

$$B_{1,7} = \frac{\left(41\omega_1^{\ 2} - 41601\right)\left(-175a^{2/3} - 17\right)}{512\omega_1^{\ 2}\omega_2^{\ 4}\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_2^{\ 2}+8\omega_2^{\ 4}\right)}$$

$$- \frac{3\sqrt{3}\left(378525 - 7654321\omega_2^{\ 2} - 271\omega_1\omega_2^{\ 3} + 341\omega_1\omega_2^{\ 5} + 13241\omega_1^{\ 2}\omega_2^{\ 6} + 235\omega_2^{\ 7}\right)}{512\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_2^{\ 2}+8\omega_2^{\ 4}\right)}$$

$$+ \frac{3\sqrt{3}a^{2/3}\left(2020074 - 1377359\omega_2^{\ 2} - 98\omega_1\omega_2^{\ 3} + 233\omega_1\omega_2^{\ 5} + 2503\omega_1^{\ 2}\omega_2^{\ 6} + 93\omega_2^{\ 7}\right)}{512\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_2^{\ 2}+8\omega_2^{\ 4}\right)}$$

$$B_{1,8} = \frac{1}{\left(-1+2\omega_1^{\ 2}\right)^2\left(-1+5\omega_1^{\ 2}\right)}\left[\frac{54\omega_1^{\ 2}\omega_2^{\ 2}}{18} + \frac{12\omega_1^{\ 2}\omega_2^{\ 2}a^{2/3}}{18}\right] + \frac{375\omega_1\omega_2 a^{2/3}}{346}$$

$$+ \frac{625\omega_1^{\ 6}\omega_2^{\ 6} + 73\omega_1\omega_2^{\ 3} + 33\omega_1^{\ 5}\omega_2^{\ 5}}{\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_1^{\ 2}+8\omega_1^{\ 4}\right)} + \frac{\left(437\omega_1^{\ 6}\omega_2^{\ 6} - 273\omega_1\omega_2^{\ 3} + 345\omega_1^{\ 5}\omega_2^{\ 5}\right)a^{2/3}}{\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_1^{\ 2}+8\omega_1^{\ 4}\right)}$$

$$B_{1,9} = \frac{3412\omega_1^{\ 3}\omega_2^{\ 3}a^{2/3}}{\left(-1+2\omega_1^{\ 2}\right)^2} + \frac{7035 - 8431\omega_1\omega_2^{\ 6} - 97\omega_1\omega_2^{\ 3} + 347\omega_1^{\ 5}\omega_2^{\ 5}}{\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_1^{\ 2}+8\omega_1^{\ 4}\right)} + \frac{173a^{2/3}\left(-41601 + 41\omega_1^{\ 2}\right)}{\omega_2^{\ 4}\left(-1+2\omega_1^{\ 2}\right)\left(-9+14\omega_1^{\ 2}+8\omega_1^{\ 4}\right)}$$

$$C_{1,1} = \frac{1}{\left(-1+2\omega_2^{\ 2}\right)^2\left(-1+5\omega_2^{\ 2}\right)^2}\left[\frac{19}{8} - \frac{5a^{2/3}}{4} + \frac{321\omega_2^{\ 2}}{24} - \frac{116\omega_2^{\ 2}a^{2/3}}{24} - \frac{321\omega_2^{\ 4}}{18} + \frac{116\omega_2^{\ 4}a^{2/3}}{18}\right]$$

$$C_{1,2} = \frac{1}{\left(-1+2\omega_2^{\ 2}\right)^2\left(-1+5\omega_2^{\ 2}\right)^2}\left[\begin{array}{l}\dfrac{a^{2/3}}{36} - \dfrac{7\omega_2^{\ 2}}{18} - \dfrac{\omega_2^{\ 2}a^{2/3}}{3} + \dfrac{9\omega_2^{\ 4}}{27} + \dfrac{4\omega_2^{\ 4}a^{2/3}}{27} \\ -\dfrac{145\omega_2^{\ 6}}{72} - \dfrac{11\omega_2^{\ 6}a^{2/3}}{36} + \omega_2^{\ 8} - \dfrac{\omega_2^{\ 8}a^{2/3}}{108}\end{array}\right]$$

$$C_{1,3} = \frac{1}{\left(-1+2\omega_2^2\right)^3 \left(-1+5\omega_2^2\right)^2} \begin{bmatrix} \dfrac{5}{2} - 2a^{2/3} - \dfrac{430\omega_2^2}{32} + \dfrac{9\omega_2^2 a^{2/3}}{32} - \omega_2^4 - \dfrac{17\omega_2^4 a^{2/3}}{2} \\ + \dfrac{103\omega_2^6}{16} - \dfrac{510\omega_2^6 a^{2/3}}{16} + \dfrac{35\omega_2^8}{2} + \dfrac{23\omega_2^8 a^{2/3}}{2} \end{bmatrix}$$

$$C_{1,4} = \frac{1}{\left(-1+2\omega_2^2\right)^2 \left(-1+5\omega_2^2\right)^2} \begin{bmatrix} \dfrac{1575a^{2/3}}{436} - \dfrac{256}{436} + \dfrac{572\omega_2^2}{436} - \dfrac{13211\omega_2^2 a^{2/3}}{436} \\ - \dfrac{5\omega_2^4}{218} + \dfrac{14285\omega_2^4 a^{2/3}}{436} - \dfrac{374\omega_2^6}{218} - \dfrac{425\omega_2^6 a^{2/3}}{218} \end{bmatrix}$$

$$C_{1,5} = \frac{1}{\left(-1+2\omega_2^2\right)^3 \left(-1+5\omega_2^2\right)^2} \begin{bmatrix} \dfrac{25a^{2/3}}{52} + \dfrac{32}{52} - \dfrac{46\omega_2^2}{52} + \dfrac{571\omega_2^2 a^{2/3}}{52} + \dfrac{12\omega_2^4}{5} \\ - \dfrac{535\omega_2^4 a^{2/3}}{25} + \dfrac{29\omega_2^6}{13} + \dfrac{1501\omega_2^6 a^{2/3}}{26} + \dfrac{283\omega_2^8 a^{2/3}}{13} \end{bmatrix}$$

$$C_{1,6} = \frac{-\left(-5+33\omega_2^2+33\omega_2^4\right)287\sqrt{3} - 287\sqrt{3}\left(2-\omega_2^2+15\omega_2^4\right)}{1024\omega_2^2\left(-9+14\omega_2^2+8\omega_2^4\right)}$$

$$C_{1,7} = \frac{-1}{512\left(9+4\omega_2^2\right)\left(-\omega_1^2+4\omega_2^2\right)\left(\omega_2^2-2\omega_2^3\right)^2} \begin{bmatrix} \sqrt{382}\omega_1^2 \begin{Bmatrix} 5-2a^{2/3}-33\omega_2^2-5\omega_2^2 a^{2/3} \\ +7\omega_2^4+9\omega_2^4 a^{2/3}+106\omega_2^6 a^{2/3}-10\omega_2^6 \end{Bmatrix} \\ +3\sqrt{3}\omega_2^2 \\ \begin{Bmatrix} -1514933+1372022a^{2/3}+61897\omega_2^2+133213\omega_2^2 a^{2/3} \\ +71303\omega_2^4+3425\omega_2^4 a^{2/3}+41151\omega_2^6 a^{2/3}+25633\omega_2^6 \end{Bmatrix} \end{bmatrix}$$

$$C_{1,8} = \frac{1}{\left(1-2\omega_2^2\right)^2 \left(-1+5\omega_2^2\right)^2} \begin{bmatrix} \dfrac{3}{2} - 2a^{2/3} + \dfrac{301\omega_2^2}{52} + \dfrac{525\omega_2^2 a^{2/3}}{32} + \dfrac{103\omega_2^6}{16} \\ - \dfrac{625\omega_2^6 a^{2/3}}{16} + \dfrac{108\omega_2^8}{109} - \dfrac{72\omega_2^8 a^{2/3}}{109} \end{bmatrix}$$

$$+ \frac{2-\omega_2^2+48\omega_2^4}{\left(-9+14\omega_2^2+8\omega_2^4\right)} - \frac{\sqrt{3}a^{2/3}\left(2-\omega_2^2+48\omega_2^4\right)}{\left(-9+14\omega_2^2+8\omega_2^4\right)}$$

$$C_{1,9} = \frac{1}{4\left(1-2\omega_2^2\right)^3 \left(-1+5\omega_2^2\right)^2} \begin{bmatrix} \dfrac{7}{2} - 2a^{2/3} + \dfrac{301\omega_2^2}{52} + \dfrac{525\omega_2^2 a^{2/3}}{32} + \dfrac{13\omega_2^4}{5} + \dfrac{25\omega_2^4 a^{2/3}}{5} \\ + \dfrac{108\omega_2^6}{33} + \dfrac{307\omega_2^6 a^{2/3}}{33} + \dfrac{33\omega_2^8}{71} - \dfrac{73\omega_2^8 a^{2/3}}{71} \end{bmatrix}$$

Appendix 3

Coefficients of $\varepsilon, A_2, W_1, W_1 e^2$ in D

$$D_2 = \frac{1}{\left(-1+2\omega_1^2\right)\left(9+4\omega_1^2\right)}\left[\frac{-85617}{16384} + \frac{9072\omega_1^2 a^{2/3}}{16384} + \left(40207796808 + 40593636714 a^{2/3}\right)\frac{\omega_1^2 \omega_2^2}{884736}\right]$$

$$+ \frac{\omega_1^2 \omega_2^2}{884736}\left[\frac{1620862}{\left(-1+2\omega_1^2\right)^2} + \frac{2a^{2/3}}{\left(-1+2\omega_1^2\right)} + \frac{2507361}{\left(1-2\omega_1^2\right)}\right]$$

$$+ \frac{\left(97354278235 + 15615339173 a^{2/3}\right)\omega_1^2 \omega_2^2}{884736\left(1-5\omega_1^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_1^2\right)\left(1-5\omega_2^2\right)\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}$$

$$+ \frac{17146183680\,\omega_1^2 \omega_2^2 a^{2/3}}{9 - 14\omega_2^2 - 8\omega_2^4}$$

$$+ \frac{1}{\left(1-5\omega_1^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_1^2\right)\left(1-5\omega_2^2\right)\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}$$

$$\times \left[\frac{787357\omega_1^4\omega_2^4}{16} + \frac{3772347\omega_1^6\omega_2^6}{8} + \frac{245443\omega_1^8\omega_2^8}{4} + a^{\frac{2}{3}}\left(\frac{241220\omega_1^4\omega_2^4}{16} + \frac{4253702\omega_1^6\omega_2^6}{8} + \frac{62731\omega_1^8\omega_2^8}{4}\right)\right]$$

$$D_3 = \frac{1}{8192\left(-1+2\omega_1^2\right)}\left[717 + 102a^{2/3} + \frac{5071}{\left(2\omega_1+\omega_2\right)\left(\omega_1+2\omega_2\right)\left(9+4\omega_2^2\right)} - \frac{\left(5734253 + 1149075a^{2/3}\right)}{\left(1-5\omega_1^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_1^2\right)\left(1-5\omega_2^2\right)\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}\right]$$

$$+ \frac{\left(12345238920 + 10291837560a^{2/3} - 800500\omega_1^4\omega_2^4 - 700300\omega_1^4\omega_2^4 a^{2/3} + 663405\omega_1^6\omega_2^6 + 95399\omega_1^6\omega_2^6 a^{2/3} + 57865\omega_1^8\omega_2^8 + 72536\omega_1^8\omega_2^8 a^{2/3}\right)}{\left(1-5\omega_1^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_1^2\right)\left(1-5\omega_2^2\right)\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}$$

$$+ \frac{3475273\omega_2^2 + 2345679 + 56244407\omega_2^2 a^{2/3} + 972081a^{2/3}}{\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)} + \frac{59719680a^{2/3}}{\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}$$

$$D_4 = \frac{1}{294912}\left[2\omega_1^2\omega_2^2\left\{\frac{1885814784a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(9-14\omega_1^2-8\omega_1^4\right)} + \frac{1345723 + 16865093a^{2/3}}{\left(1-7\omega_1^2+10\omega_1^4\right)\left(1-7\omega_2^2+10\omega_2^4\right)} + \frac{5864788}{\left(1-2\omega_1^2\right)} - \frac{75357 + 110808a^{2/3}}{\left(-1+2\omega_1^2\right)^2\left(4\omega_1^2-\omega_2^2\right)}\right\}\right.$$

$$\left. + 243\left\{\frac{63417}{\left(9-14\omega_1^2-8\omega_1^4\right)} + \frac{38543 - 5087a^{2/3}}{\left(9-14\omega_1^2-8\omega_1^4\right)} + \frac{25799a^{2/3}}{\left(9-14\omega_1^2-8\omega_1^4\right)} + \frac{4357 + 351a^{2/3}}{\left(-1+2\omega_1^2\right)^2} + \frac{63478 - 5001a^{2/3}}{\left(1-2\omega_1^2\right)^2}\right\}\right]$$

$$D_5 = \frac{1}{49152}\left[\begin{array}{l}\omega_1^2\omega_2^2\left\{\begin{array}{l}\dfrac{107750+121596a^{2/3}}{\left(1-2\omega_1^2\right)^2}+\dfrac{23125+16124a^{2/3}}{\left(1-2\omega_1^2\right)\left(4\omega_1^2-\omega_2^2\right)}+\dfrac{447897500+100a^{2/3}}{\left(-1+2\omega_1^2\right)\left(9+4\omega_2^2\right)}+\dfrac{95256500+760a^{2/3}}{\left(9-14\omega_2^2-8\omega_2^4\right)}\\[4mm]+\dfrac{3564256156+2711833700a^{2/3}}{\left(1-5\omega_1^2\right)\left(-9+14\omega_1^2+8\omega_1^4\right)\left(1-5\omega_2^2\right)\left(-9+14\omega_2^2+8\omega_2^4\right)}-\dfrac{594542592a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(-9+14\omega_2^2+8\omega_2^4\right)}\end{array}\right\}\\[10mm]+\dfrac{\begin{array}{l}23573245\omega_1^4\omega_2^4+3136215299\omega_1^4\omega_2^4a^{2/3}+2357243593\omega_1^6\omega_2^6\\+46954801463\omega_1^6\omega_2^6a^{2/3}+745273252\omega_1^8\omega_2^8+2989676636\omega_1^8\omega_2^8a^{2/3}\end{array}}{\left(1-5\omega_1^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_1^2\right)\left(1-5\omega_2^2\right)\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}\\[8mm]+9\left\{\begin{array}{l}\dfrac{3419+3007a^{2/3}}{\left(-9+14\omega_1^2+8\omega_1^4\right)}-\dfrac{2347+110a^{2/3}}{\left(1-2\omega_1^2\right)}\\[4mm]-\dfrac{30450687+a^{2/3}}{\left(1-5\omega_1^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_1^2\right)\left(1-5\omega_2^2\right)\left(-1+2\omega_2^2\right)\left(9+4\omega_2^2\right)}\end{array}\right\}\end{array}\right]$$

$$D_6 = \frac{1}{82944\sqrt{3}}\left[2\omega_1^2\omega_2^2\left\{\begin{array}{l}\dfrac{-578-160a^{2/3}}{\left(1-2\omega_1^2\right)^2}+\dfrac{82544+11355a^{2/3}}{\left(1-2\omega_1^2\right)^2}+\dfrac{91000060+445700a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(-9+14\omega_2^2+8\omega_2^4\right)}\\[4mm]+2988\left\{\dfrac{63-11a^{2/3}}{\left(1-2\omega_1^2\right)^2}+\dfrac{4+3a^{2/3}}{\left(-9+14\omega_2^2+8\omega_2^4\right)}\right\}\end{array}\right\}\right]$$

$$D_7 = \frac{1}{110592\sqrt{3}}\left[\begin{array}{l}2\omega_1^2\omega_2^2\left\{\begin{array}{l}\dfrac{-492a^{2/3}}{\left(1-2\omega_1^2\right)^2}+\dfrac{48077+322887a^{2/3}}{\left(-1+2\omega_1^2\right)}+\dfrac{103680a^{2/3}}{\left(9+4\omega_2^2\right)\left(1-2\omega_2^2\right)^2}\\[4mm]-\dfrac{116702208a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(-1+2\omega_2^2\right)^2\left(9+4\omega_2^2\right)}-\dfrac{157589542+88588250a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(9+4\omega_2^2\right)}\\[4mm]+\dfrac{62519040a^{2/3}}{-9+14\omega_2^2+8\omega_2^4}+\dfrac{72593+798319a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(-1+2\omega_1^2\right)^2\left(9+4\omega_2^2\right)}\\[4mm]+\dfrac{13893120a^{2/3}}{\left(9+4\omega_2^2\right)^2\left(1-2\omega_2^2\right)}-\dfrac{61774+3121a^{2/3}}{\left(4\omega_1^2-\omega_2^2\right)\left(-1+2\omega_1^2\right)}\end{array}\right\}\\[14mm]+27\left\{\dfrac{5000+904a^{2/3}}{\left(-1+2\omega_1^2\right)^2}+\dfrac{347315-225158a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(-1+2\omega_1^2\right)^2}-\dfrac{553576+20451a^{2/3}}{\left(\omega_1^2-4\omega_2^2\right)\left(-1+2\omega_1^2\right)^2}+\dfrac{5000+904a^{2/3}}{9-14\omega_2^2-8\omega_2^4}\right\}\end{array}\right]$$

$$D_8 = \frac{1}{8912\sqrt{3}}\left[\begin{array}{l}775\left\{\dfrac{1389357}{\left(1-2\omega_2^2\right)^2}+\dfrac{34523\omega_2^2}{\left(1-2\omega_1^2\right)^2}+\dfrac{72592}{\omega_1^2-4\omega_2^2}+\dfrac{62519245a^{2/3}}{\left(-9+14\omega_2^2+8\omega_2^4\right)}+\dfrac{5007}{\left(9-14\omega_2^2+8\omega_2^4\right)}\right\}\\[6mm]+2\omega_1^2\omega_2^2\left\{\dfrac{13873257}{\left(9+4\omega_2^2\right)\left(1-2\omega_2^2\right)}+\dfrac{103687}{\left(9+4\omega_2^2\right)\left(1-2\omega_2^2\right)}+\dfrac{62579245a^{2/3}}{\left(-9+14\omega_2^2+8\omega_2^4\right)}+\dfrac{48077}{\left(1-2\omega_1^2\right)}\right\}\\[6mm]+\dfrac{\left(235757\omega_1^4\omega_2^4+3754253\omega_1^6\omega_2^6+75654378\omega_1^8\omega_2^8\right)a^{2/3}}{\left(1-5\omega_1^2\right)\left(-9+14\omega_1^2+8\omega_1^4\right)}\end{array}\right]$$

$$D_9 = \frac{1}{7345\sqrt{3}} \left[\begin{array}{c} \dfrac{\begin{array}{c} 14522245\omega_1^{\,4}\omega_2^{\,4} + 345623578\omega_1^{\,4}\omega_2^{\,4}a^{2/3} + 2357243593\omega_1^{\,6}\omega_2^{\,6} \\ +464562145\omega_1^{\,6}\omega_2^{\,6}a^{2/3} + 65855236\omega_1^{\,8}\omega_2^{\,8} + 7895456\omega_1^{\,8}\omega_2^{\,8}a^{2/3} \end{array}}{\left(1-5\omega_1^{\,2}\right)\left(-1+2\omega_1^{\,2}\right)^2\left(9+4\omega_1^{\,2}\right)\left(1-5\omega_2^{\,2}\right)\left(-1+2\omega_2^{\,2}\right)\left(9+4\omega_2^{\,2}\right)} \\ +2\omega_1^{\,2}\omega_2^{\,2}\left\{ \begin{array}{c} \dfrac{7856+456a^{2/3}}{\left(1-2\omega_1^{\,2}\right)^2} + \dfrac{845654+45628a^{2/3}}{\left(1-2\omega_1^{\,2}\right)^2} \\ +\dfrac{9145658+4456258a^{2/3}}{\left(\omega_1^{\,2}-4\omega_2^{\,2}\right)\left(-9+14\omega_2^{\,2}+8\omega_2^{\,4}\right)} \end{array} \right\} \end{array} \right]$$