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Topological Properties of Distance Space

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Abstract In this paper, we intended to explain through some examples the usefulness and necessity of studying the topological properties of weaker forms of distance spaces.

Keywords: Distance space, continuity.

1 Introduction

Many decades ago topology and topological properties were used in many areas. Certainly the areas includes the algebraic, general, geometric and set-theoretic facets of topology as well as areas of interactions between topology and other mathematical disciplines, e.g. topological algebra, topological dynamics, functional analysis, abstract spaces, category theory and so on.But it is limited to pure and applied mathematics. In recent years, topology and topological properties are expanded in a wide range and plays very important role in pure and applied mathematics but not limited to these subjects. Since the roles of various aspects of topology continue to change, the non-specific delineation of topics serves to reflect the current state of research in Physics, Computer science, Biology and so many.

In recent years, many authors prove various fixed point theorems in certain spaces but very few authors present the role of topology and topological properties in a certain spaces. We would like to discuss and present some topology and topological aspects of certain spaces. For that we choose metric fixed point theory as a tool. Thus Frechet [1] selected, what might have appeared to him as the three most important properties that a *distance function* must satisfy. Standard generalizations of metric space like topological space or uniform space dispense with the notion of distance altogether and study the resulting *fluid* geometry as opposed to the *rigid* geometry of the metric space. There is yet another way to generalize the notion of a metric space: by retaining the notion of a metric and yet weakening the axioms imposed on a metric. Such studies have been initiated by Chittinden [2], Fisher [3], Frink[4], Wilson [5] and others and have received a stimulus from the work of computer scientists such as Pascal Hitzler [6], who found that their studies of computer languages and programs need weakened notions of a metric such as partial metric ,dislocated metric, quasi dislocated metric etc. This manuscript is meant to study how weakening of the notion of a metric affects the topology that is induced by the weak metric. The process by which a metric generates a topology often needs to be suitably modified if a weakened form of a metric is to yield a topology. Moreover we should not lose sight of the fact that topology is essentially the language best suited for the description and study of continuity. So the general thrust of any attempt at a generalization of the concept of metric should be to examine how the intuitive notion of continuity, applied to that weak metric is reflected in the associated topology.

It was more than a century ago, in 1906, that Maurice Frechet introduced the notion of metric space in his Thesis *Surquelques points du calcul functionnel*[1]. It was about a century ago, in 1914, that Felix Hausdorff published his study of metric and topological spaces in *Grundzuge der Mengenlehre*[7]. Though the concept of distance distills three natural conditions of the usual Euclidean distance in the physical 3-space, this could be applied to much more complex objects. We can talk about distance between curves defined on a common domain and about distance between closed and bounded subsets of a metric space and so on. Moreover some very natural properties enjoyed by the Euclidean distance such as *convexity* are ignored in the definition of a metric.

Definition 1.1. A metric space (X, d) is called convex if given a positive number ϵ and distinct points x, y with distance $d(x, y) = r > \epsilon$, it is possible to find a point z such that $x \neq z \neq y,\ d(x, y) = d(x, z) + d(z, y)$ and $d(x, z) = \epsilon$.

We recall that a metric on a nonempty set X is a nonnegative real valued function d on $X \times X$ satisfying,

$$\begin{aligned} &\text{(i)}\ \ d(x,y) = 0 \Leftrightarrow x = y \\ &\text{(ii)}\ \ d(x,y) = d(y,x) \\ &\text{(iii)}\ \ d(x,y) \leq d(x,z) + d(z,y) \text{ for all } x,y,z \text{ in } X. \end{aligned}$$

If d is a metric on X , the pair (X, d) is called a metric space (ii), is called symmetry while (iii) is referred to as the triangular inequality. The set $B_r(x) = \{y/y \in X \text{ and } d(x, y) < r\}$ is called open ball centered at x while $B_r[x] = \{y/y \in X \text{ and } d(x, y) < r\}$ is called closed ball.

(i) ensures that balls are nonempty and (ii)& (iii) together ensure that open balls are open, closed balls are closed and that metric spaces are Hausdorff spaces. We can weaken or dispense with (i) or (ii) or (iii) in the definition of a metric at the cost of these pleasant consequences. But there are metric like functions that do not enjoy (i) or (ii) or (iii) in full force, but are useful in the study of several practical problems. In addition to this utilitarian view, there is the platonic pure mathematical need to consider the logical consequences of weakening the defining conditions of a metric. As such, we are naturally motivated to study topological and geometrical aspects of various weakened forms of a metric space.

2 Topology of Distance Spaces

Definition 2.1. Let X be a non-empty set and a mapping $d : X \times X \to R^+$ is called a distance space and some related conditions that may or may not satisfy are defined as follows:
d_1 : Self distances are zero:$d(x, x) = 0; \forall x \in X$
d_2 : Distances are symmetric:$d(x, y) = d(y, x); \forall x, y \in X$
$d_3 : d(x, y) = 0 \Rightarrow x = y$
$d_4 : d(x, y) > 0 \Rightarrow x \neq y$
d_5 : Triangle inequality:$d(x, y) \leq d(x, z) + d(z, y); \forall x, y, z \in X$
d_6 : Ultrametric property:$d(x, y) \leq max\{d(x, z), d(z, y)\}; \forall x, y, z \in X$
$d_7 : d(x, y) \leq \gamma; d(y, z) \leq \gamma \Rightarrow d(x, z) \leq \gamma \forall x, y, z \in X$
d_8 : Quadrilateral property:If x, y, u, v are distinct $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$
d_9 : Left triangle inequality:$d(x, y) \leq d(z, x) + d(z, y); \forall x, y, z \in X$
d_{10} : Right triangle inequality:$d(x, y) \leq d(x, z) + d(y, z); \forall x, y, z \in X$

Definition 2.2. If $x \in X$ and $\epsilon > 0$, the set
(a). $B^r_\epsilon(x) = \{y/y \in X \text{ and } d(x, y) < r\}$ is called the right (open)ball centered on x with radius ϵ.
(b). $V^r_\epsilon(x) = B^r_\epsilon(x) \cup \{x\}$ is called the right inclusive (open) ball centered on x with radius ϵ.
(c). $B^l_\epsilon(x) = \{y/y \in X \text{ and } d(y, x) < r\}$ is called the left (open)ball centered on x with radius ϵ.
(b). $V^l_\epsilon(x) = B^r_\epsilon(x) \cup \{x\}$ is called the left inclusive (open)ball centered on x with radius ϵ.
(e). $B_\epsilon(x) = B^r_\epsilon(x) \cap B^l_\epsilon(x)$ is called the bilateral(open) ball centered on x with radius ϵ.
(f). $V_\epsilon(x) = B_\epsilon(x) \cup \{x\}$ is called the bilateral inclusive (open) ball centered on x with radius ϵ , and x is called the center while ϵ is called the radius of the ball.

Note that radii as well as centers are not necessarily unique. For example, in N with the discrete metric, $B_3(2) = B_{100}(23) = N$. Some of the similar argument can be found in [9-14].

Remark 2.3. (1) If d is symmetric then $B_\epsilon(x) = B^r_\epsilon(x) = B^l_\epsilon(x)$ and $V_\epsilon(x) = V^r_\epsilon(x) = V^l_\epsilon(x)$. (2) If self distances are zero then $B_\epsilon(x) = V_\epsilon(x), B^l_\epsilon(x) = V^l_\epsilon(x)$ and $B^r_\epsilon(x) = C$.

Definition 2.4. If $A \subset X, x \in A$, is called a right interior point of A , and A is called a right neighborhood (simply nbd) of x if there exists $\epsilon > 0$ such that $V^r_\epsilon(x) \subseteq A$.

Corresponding left notion is defined similarly.
Result: Let (X, d) be a distance space. If d satisfies triangular inequality then right balls(left balls) are open.

Proposition 2.5. *If d satisfies triangular inequality* (d_5) *or ultra metric property* (d_6) *then* $V^r_\epsilon(x)(V^l_\epsilon(x))$ *is a right (left) nbd of every point in it.*

Proof. Assume (d_5).If $y \in V^r_\epsilon(x)$ then $\delta = \epsilon - d(x, y) > 0$. If $z \in V^r_\delta(y)$ then $d(y, z) < \delta = \epsilon - d(x, y)$ so that $d(x, z) \leq d(x, y) + d(y, z) < \epsilon$.
Hence $z \in V^r_\epsilon(x)$. This implies $V^r_\delta(x) \subseteq V^r_\epsilon(x)$.
Since $(d_6) \Rightarrow (d_5)$ □

Remark 2.6. If d satisfies the left triangular inequality
$d_5^{'} : d(x, y) \leq d(z, x) + d(z, y)$ (or)
the left ultra metric property
$d_6^{'} : d(x, y) \leq max\{d(z, x), d(z, y)\}$
then $V^l_\epsilon(x)$ is a right nbd of every point in it.

A simple method for constructing distance functions [8]:

Proposition 2.7. *Let* ρ *be a metric on* $X, A \subseteq X$ *and* $f : R^+ \to R^+$. *Define* $d : A \times A \to R^+$ *by* $d(x, y) = f(\rho(x, y))$ *for* $x, y \in A$. *Then*
(a) *d is a symmetric distance function on A*
(b) *If f is monotonically increasing and subadditive, then d satisfies triangle inequality* d_5.
(c) *If* $f^{-1}(0) = \{0\}$ *then* $d(x, y) = 0 \Leftrightarrow x = y$

Proof. (a) is clear.
(b) Let $x, y, z \in A, \rho(x, y) = a,\ \rho(y, z) = b$ and $\rho(z, x) = c$. By the triangle inequality $c \leq a + b$. Since f is increasing and subadditive,
$f(c) \leq f(a + b) \leq f(a) + f(b)$
Hence $d(x, z) = f(c) \leq f(a) + f(b) = d(x, y) + d(y, z)$.
(c) $d(x, y) = 0 \Leftrightarrow f(\rho(x, y)) = 0 \Leftrightarrow \rho(x, y) = 0 \Leftrightarrow x = y$ □

Example 2.7(a):

1. Define $\rho(x, y) = |x - y|$ for x, y in R and $f(x) = x^2$ for $x \in R^+, d(x, y) = (x - y)^2$ defines a distance function that satisfies $d(x, y) = 0 \Leftrightarrow x = y$.
   If $x > 0, d(x, 0) = d(0, -x) = x^2$ and $d(x, -x) = 4x^2$.
   Thus $d(x, -x) \nleq d(x, 0) + d(0, -x)$ so that d does not satisfy triangle inequality.
2. Define $\rho(x, y) = |x - y|$ for x, y in R and $f(x) = e^x$ for $x \in R^+, d(x, y) = e^{|x-y|}$ defines a distance function R and $d(x, y) > 0$.

In the following theorem we present a few more methods to create distance functions.

Proposition 2.8. *Let* f, g *be nonnegative real valued functions on a set* X . *Define* $d(x, y) = f(x) + g(y)$ *for* $x, y \in X$.

1. $d(x, y) = d(y, x) \forall x, y$ *if and only if* $(f - g)$ *is a constant function.*
2. *d satisfies triangular inequality.*

Proof. $(1).d(x, y) = d(y, x)$
$\Leftrightarrow f(x) + g(y) = f(y) + g(x)$
$\Leftrightarrow f(x) - g(x) = f(y) - g(y)$
$\Leftrightarrow f - g$ is constant function.
$(2).d(x, y) + d(y, z) = f(x) + g(y) + f(y) + g(z) \geq f(x) + g(z) = d(x, z)$ □

Proposition 2.9. *Let* $X \subseteq R^+$ *and* $A = X - X = \{a - b/a \in X, b \in X\}$ *and* $d(x, y) = f(x - y)$ *where* $f : A \to R^+$. *If* $f^{-1}(0) = \{0\}$ *and f is even and subadditive then d is a metric.*

Proof. $d(x, y) = 0 \Leftrightarrow f(x - y) = 0 \Leftrightarrow x = y$
$d(x, y) = d(y, x) \Leftrightarrow f(x - y) = f(y - x) \Leftrightarrow f$ is even.
$d(x, y) + d(y, z) = f(x - y) + f(y - z) \geq f(x - y + y - z) = f(x - z) = d(x, z)$ □

Proposition 2.10. *Let $f : X \to R^+$ and $g : X \to R^+$ be functions such that $inf\{f(x)/x \in X\} \geq sup\{g(y)/y \in X\}$. Define $d(x, y) = f(x) - g(y)$. Then d satisfies the triangle inequality. Further d is symmetric if and only if $f + g$ is constant.*

Proof. $d(x, y) \geq 0$, by hypothesis $d(x, y) = d(y, x) \Leftrightarrow f(x) - g(y) = f(y) - g(x)$
$\Leftrightarrow (f + g)x = (f + g)y$
$\Leftrightarrow f + g$ is constant.
For any x, y, z in X;
$d(x, y) + d(y, z) - d(x, z)$
$= f(x) - g(y) + f(y) - g(z) - f(x) = g(z)$
$= f(y) - g(y) \geq 0$ by hypothesis. □

Proposition 2.11. *For any $f : X \to R^+, d(x, y) = |f(x) - f(y)|$ defines a metric if and only if f is one-one.*

Proof. Clearly $d(x, y) \geq 0$ and $d(x, x) = 0$
$d(x, y) = 0 \Leftrightarrow |f(x) - f(y)| = 0 \Leftrightarrow f(x) = f(y)$
then if f is one-one, $d(x, y) = 0 \Rightarrow x = y$
Conversely if d is a metric and $f(x) = f(y)$ then $d(x, y) = 0$, hence $x = y$.
Since $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)|$, triangle inequality holds. □

Examples 2.11a:

1. $d(x, y) = (x - a)^2 + (y - b)^2$ for $x, y \in R$
   Clearly $d(x, y) \geq 0$ for all x, y and d satisfies triangle inequality.
   d is symmetric iff $(x - a)^2 - (x - b)^2 = (y - a)^2 - (y - b)^2 \forall x, y$
   $\Leftrightarrow (2x - a - b)(b - a) = (2y - a - b)(b - a) \forall x, y$
   $\Leftrightarrow 2(x - y)(b - a) = 0 \forall x, y$
   $\Leftrightarrow a = b$
2. Define $d(x, y) = (x - y)^2$ for $x, y \in R$
   Clearly $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$ and $d(x, y) = d(y, x)$.
   Triangular inequality does not hold since $d(-1, 2) = 9 > d(-1, 1) + d(1, 2)$.
   We shall now verify that balls are open in (X, d).
   For all $x \in R$ and $\epsilon > 0, B_\epsilon(x) = (x - \sqrt{\epsilon}, x + \sqrt{\epsilon})$.
   So that if $|x - y| < \sqrt{\epsilon}$ and $|y - z| < \sqrt{\epsilon} - |x - y|$ then $|x - z| \leq |x - y| + |y - z| < \sqrt{\epsilon}$.
   Hence $d(x, z) < \epsilon$.
   Thus the triangular inequality is not necessary for balls to be open.
3. Define d on R by

$$d(x, y) = \begin{cases} 1, \text{ if } x = y \\ 5, \text{if } 0 < |x - y| < 1 \\ |x - y|, \text{if } |x - y| \geq 1 \end{cases} \tag{2.1}$$

   $d(x, y) = d(y, x)$
   $d(x, y) > 0$ for all x, y and triangular inequality does not hold since
   $d(4, 4.5) = 5, d(4, 5.5) = 1.5$ and $d(5.5, 4.5) = 1$
   It is easy to verify that ,

$$B_\epsilon(1) = \begin{cases} \phi, \text{ if } 0 < \epsilon \leq 1 \\ [2, 1 + \epsilon) \cup (1 - \epsilon, 0], \text{if } 0 < \epsilon \leq 5 \\ (1 - \epsilon, 1 + \epsilon), \text{if } \epsilon > 5 \end{cases} \tag{2.2}$$

$$B_\delta(0) = \begin{cases} \phi, \text{ if } 0 < \delta \leq 1 \\ (-\delta, -1] \cup [1, \delta) \cup \{0\}, \text{if } 0 < \delta \leq 5 \\ (-\delta, \delta), \text{if } \delta > 5 \end{cases} \tag{2.3}$$

When $\epsilon = 1, B_4(1) = [2, 5) \cup (-3, 0] \cup \{1\}$ so that $0 \in B_4(1)$.
However there does not exist $\delta > 0$ such that $\phi \neq B_\delta(0) \subseteq B_4(1)$.

Proposition 2.12. *Let (X, d) be a distance space. If d satisfies the triangle inequality the set of all right inclusive balls $\{V^r_\epsilon(x)/x \in X, \epsilon > 0\}$ is a base for a topology on X.*

Proof. Clearly every $x \in X$ belongs to $V^r_\epsilon(x)$ for every $\epsilon > 0$.
If $x_1, x_2 \in X, \epsilon_1, \epsilon_2 > 0$ and $z \in V^r_{\epsilon_1}(x_1) \cap V^r_{\epsilon_2}(x_2)$,
$d(x_i, z) < \epsilon_i$ for $i = 1, 2$. Let $\epsilon = min\{\epsilon_1 - d(x_1, z), \epsilon_2 - d(x_2, z)\}$.
If $y \in V^r_\epsilon(z), d(z, y) < \epsilon \leq \epsilon_i - d(x_i, z)$ for $i = 1, 2$.
$\Rightarrow d(x_i, z) + d(z, y) < \epsilon_i$ for $i = 1, 2$.
$\Rightarrow d(x_i, y) \leq d(x_i, z) + d(z, y) < \epsilon_i$ for $i = 1, 2$.
$\Rightarrow y \in V^r_{\epsilon_1}(x_1) \cap V^r_{\epsilon_2}(x_2)$
$\Rightarrow V^r_\epsilon(z) \subseteq V^r_{\epsilon_1}(x_1) \cap V^r_{\epsilon_2}(x_2)$
This proves that the set of right inclusive balls is a base for a topology on X. □

Definition 2.13. The topology induced by the set of right inclusive balls is called the right topology induced by d on X and is denoted by $\Im^d_r$ or simply $\Im_r$.

Remark 2.14.
1. The set of left inclusive balls induces a topology. We call this , the left topology induced by d on X and denote this by $\Im^d_l$ or simply $\Im_l$.
2. The above proof suggests that the triangular inequality is not in fact essential for one sided inclusive balls to induce a topology but it is sufficient for one sided balls to be open.
3. If d is symmetric then $V^r_\epsilon(x) = V^l_\epsilon(x)$ and hence $\Im_r = \Im_l$.

Theorem 2.15. *Let (X, d) be a distance space satisfying $d_4, d_5, d_{10}(d_9)$. Then $\Im_r(\Im_l)$ is Haussdorff and it is first countable.*

Proof. Let $x \neq y \in X$.
Suppose $V^r_{\frac{\epsilon}{2}}(x) \cap V^r_{\frac{\epsilon}{2}}(y) \neq 0$
$\Rightarrow z \in V^r_{\frac{\epsilon}{2}}(x)$ and $z \in V^r_{\frac{\epsilon}{2}}(y)$
$\Rightarrow d(x, z) < \frac{\epsilon}{2}; d(y, z) < \frac{\epsilon}{2}$
$\Rightarrow d(x, z) < \epsilon$. A contradiction.
Since $x \neq y$ implies $d(x, y) > 0$. □

Theorem 2.16. *Let (X, d) be a distance space, d satisfies the triangle inequality. Then the distance function D defined by $D(x, y) = d(x, y) + d(y, x)$ induces the topology $\Im_D = \Im_r \cap \Im_l$.*

Proof. Clearly D is symmetric and satisfies the triangle inequality and hence induces a topology $\Im_D$.
Further if $x \in X$ and $\epsilon > 0$ then $V_\epsilon(x) \subseteq V^r_\epsilon(x) \cap V^l_\epsilon(x) \subseteq V_{2\epsilon}(x)$.
where $V_\epsilon(x)$ is the inclusive ball with center x w.r.t D while $V^l_\epsilon(x)$ and $V^r_\epsilon(x)$ are respectively the right and left inclusive balls w.r.t d.
It is thus sufficient to show that the collection $B = \{V^r_\epsilon(x) \cap V^l_\eta(y)/x, y \in X, \epsilon > 0, \eta > 0\}$ is a base for $\Im_r \cap \Im_l$.
Clearly if $x \in X$ and $\epsilon > 0$, then $V^r_\epsilon(x) \cap V^l_\epsilon(x)$.
Now let $x_1, y_1, x_2, y_2 \in X$ and $\epsilon_1, \eta_1, \epsilon_2, \eta_2 > R^+$
$U = \{V^r_{\epsilon_1}(x_1) \cap V^l_{\eta_1}(y_1)\} \cap \{V^r_{\epsilon_2}(x_2) \cap V^l_{\eta_2}(y_2)\}$ and $z \in U$ then $z \in V^r_{\epsilon_1}(x_1) \cap V^r_{\epsilon_2}(x_2)$ and $z \in V^l_{\eta_1}(y_1) \cap V^l_{\eta_2}(y_2)$.
$\Rightarrow$ there exist $\delta_1, \delta_2 > 0 \ni V^r_{\delta_1}(z) \subseteq V^r_{\epsilon_1}(x_1) \cap V^r_{\epsilon_2}(x_2)$ and $V^l_{\delta_2}(z) \subseteq V^l_{\eta_1}(y_1) \cap V^l_{\eta_2}(y_2)$.
If $\delta = min\{\delta_1, \delta_2\}$ then $\delta > 0$ and $z \in V^r_\delta(z) \cap V^l_\delta(z) \subseteq U$
Hence B is a base for $\Im_r \cap \Im_l$.
Hence $\Im_d = \Im_r \cap \Im_l$. □

Examples 2.17:
Example 2.17(i): Let $X = (0, \infty)$ and

$$d(x,y) = \begin{cases} x+y, \text{ if } x<y \\ x-y, \text{if } x \geq y \end{cases} \tag{2.4}$$

Let $x \in X$ and $\epsilon > 0$.
If $x < \frac{\epsilon}{2}$ then $x + y < \epsilon \Leftrightarrow y \in (x, \epsilon - x)$
Also if $0 < y \leq x$ then $x - y < \epsilon \Leftrightarrow x - \epsilon < 0 < y$
Thus if $0 < x < \frac{\epsilon}{2}$ then $V^r_\epsilon(x) = (0, x] \cup (x, \epsilon - x)....(a)$
If $\frac{\epsilon}{2} \leq x < \epsilon$ then $\epsilon - x \leq x$ so $x - y$ then
Further if $y \leq x$ then $0 \leq x - y \leq x < \epsilon$.
Thus if $\frac{\epsilon}{2} \leq x < \epsilon$ then $V^r_\epsilon(x) = (0, x]....(b)$
If $x \geq \epsilon, x + y > \epsilon$ while $y \leq x, x - y < \epsilon \Leftrightarrow x - \epsilon < y \leq x$
So that $V^r_\epsilon(x) = (x - \epsilon, x]$ if $x \geq \epsilon$.
Hence

$$V^r_\epsilon(x) = \begin{cases} (0, \epsilon - x), \text{ if } x < \frac{\epsilon}{2} \\ (0, \epsilon], \text{ if } \frac{\epsilon}{2} \leq x < \epsilon \\ (x - \epsilon, x] \text{if } x \geq \epsilon \end{cases} \tag{2.5}$$

Clearly $x \in V^r_\epsilon(x)$.
If $0 < a < x < b$ and $0 < \epsilon < x - a$ then $a < x - \epsilon < x < b$
So $V^r_\epsilon(x) = (x - \epsilon, x] \subseteq (a, b)$.
Hence $(a, b) \in \Im_r \forall b > a > 0$.
Also $(a, b] \in \Im_r \forall b > a > 0$.Since

1. If $a < x < b$ and $0 < \epsilon < x - a$ then $V^r_\epsilon(x) \subseteq (a, b)$ while
2. If $0 < \epsilon < b - a, a < b - \epsilon < b$ so $V^b_\epsilon = (b - \epsilon, b] \subseteq (a, b]$.

If $0 < y < x < \frac{\epsilon}{2}$ then $y + x < \epsilon \Leftrightarrow 0 < y < x < \epsilon - x \Leftrightarrow y \in (0, x)$ and $x < \frac{\epsilon}{2}$.
Further if $0 < x \leq y$ then $y < x < \epsilon \Leftrightarrow x - y < x + \epsilon \Leftrightarrow y \in [x, x + \epsilon]$.
Thus if $0 < x < \frac{\epsilon}{2}, d(y, x) < \epsilon \Leftrightarrow y \in (0, x + \epsilon)$.
If $\frac{\epsilon}{2} \leq x < \epsilon$ then $y < x < y + x < \epsilon \Leftrightarrow 0 < y < \epsilon - x \leq x$.
If $x \leq y, 0 \leq y - x < \epsilon \Leftrightarrow y < x + \epsilon$.
Thus if $\frac{\epsilon}{2} \leq x < \epsilon, d(y, x) < \epsilon \Leftrightarrow y \in (0, \epsilon - x) \cup [x, x + \epsilon]$.
If $x \geq \epsilon$ then $x + y > \epsilon$ when $0 < y < x$.
If $x \leq y, y - x < \epsilon$ when $y < x + \epsilon$.
Thus if $x \geq \epsilon, d(y, x) < \epsilon \Leftrightarrow y \in [x, x + \epsilon)$.
Hence

$$V^l_\epsilon(x) = \begin{cases} (0, x + \epsilon), \text{ if } 0 < x < \frac{\epsilon}{2} \\ (0, \epsilon - x] \cap [x, x + \epsilon), \text{ if } \frac{\epsilon}{2} \leq x < \epsilon \\ [x, x + \epsilon], \text{ if } x \geq \epsilon \end{cases} \tag{2.6}$$

If $0 < a < b, (a, b) \in \Im_l$ since $\forall x \in (a, b)$ and $0 < \epsilon < min\{b - x, x\}$ then $V^l_\epsilon(x) = [x, x + \epsilon) \subseteq (a, b)$.
However $(a, b] \notin \Im_l$ since $V^l_\epsilon(b) \nsubseteq (a, b]$ for any $\epsilon > 0$.
Thus $\Im_r \neq \Im_l$.
Further $D(x, y) = d(x, y) + d(y, x) = 2y$ for all x, y and $V_\epsilon(x) = \{y/D(x, y) < \epsilon\} = (0, \frac{\epsilon}{2})$.
Example 2.17(ii): Define $d(x, y) = x + y$ if $x + y > 0$ and 0 if $x + y \leq 0$.
Clearly $d(x, y) = 0 \Leftrightarrow x \leq -y$
$d(x, y) = d(y, x)$
If $x > z > 0$ and $y = -x, y + z = -x + z > 0$
$d(x, y) = d(x, -x) = 0, d(y, z) = d(-x, z) = 0$.
Thus $d(x, y) = d(y, z) = 0$ but $d(x, z) = x + z > 0$
so triangle inequality does not hold.
For that $B_\epsilon(x) = (-\infty, -x] \cap (-x, \epsilon - x) = (-\infty, \epsilon - x)$.

Example 2.17(iii): $d(x,y) = |x|$ for $x, y \in R$
Clearly $d(x,y) = d(y,x)$ if and only if $y = \pm x$.
Further d satisfies triangle inequality.

$$B_r^{\epsilon}(x) = \{y/d(x,y) < \epsilon\} = \{y/|x| < \epsilon\} = \begin{cases} \phi, & \text{if } \epsilon \leq |x| \\ R, & \text{if } |x| < \epsilon \end{cases} \tag{2.7}$$

Thus

$$V_{\epsilon}^{r}(x) = \begin{cases} \{x\}, & \text{if } \epsilon \leq |x| \\ R, & \text{if } |x| < \epsilon \end{cases} \tag{2.8}$$

Hence $\Im_r = P(X)$.
$B_{\epsilon}^{l}(x) = \{y/d(x,y) = |x| < \epsilon\} = (-\epsilon, \epsilon) \forall x$
So that $V_{\epsilon}^{l}(x) = (-\epsilon, \epsilon) \cup \{x\}$.
Thus $A \subset R$ is left open if and only if $\exists \epsilon > 0 \ni (-\epsilon, \epsilon) \subseteq A$
Hence $\Im_l = \{A/\exists \epsilon > 0 \ni (-\epsilon, \epsilon) \subseteq A\} \cup \{\phi\}$.
Clearly $\Im_r \neq \Im_l$.

3 Convergence of Sequences

In a metric space (X, d) sequential convergence w.r.t d plays an important role.
For example,

1. Constant sequences are convergent.
2. Limits are unique
3. Accumulation points of a set are precisely the limits of nonconstant sequences from the set and
4. Every convergent sequence is a Cauchy sequence.

One natural question is whether these hold good in arbitrary distance spaces.

Definition 3.1. Let (X, d) be a distance space ,$\{x_n\}$ is a sequence in X and $x \in X$. We say that $\{x_n\}$ right converges to x if $\lim d(x_n, x) = 0$ and left converges to x if $\lim d(x, x_n) = 0$.

Clearly $\{x_n\}$ converges to x w.r.t D iff $\{x_n\}$ right as well as left converges to x w.r.t d.
$\lim d(x_n, x) = 0 = \lim d(x_n, y) \Rightarrow d(x, y) = 0$ and finally if $x \in X$ and $x_n = x \forall n$, $\lim d(x_n, x) = 0$ if and only if $d(x, x) = 0$.

The proof of the following theorem is routine and hence is stated without proof.

Theorem 3.2. *Let (X, d) be a distance space where d satisfies d_2, d_4 and d_5(or d_7). Then*

1. *If $\lim d(x_n, x) = 0 = \lim d(x_n, y)$ then $d(x, y) = 0$*
2. *$A \subset X$ and $x \in X$ then $x \in \overline{A}$ iff either $x \in A$ or there exists a sequence $\{x_n\}$ in A such that $\lim d(x_n, x) = 0$*
3. *Every convergent sequence is a cauchy sequence.*

Remark 3.3. If we define $x\theta y$ iff $d(x, y) = 0$ then θ is an equivalence relation and (1),(2) and (3) hold for the metric space X/θ of equivalence classes modulo θ.

Example 3.4: Define $d(x, y) = |x|$ if $x, y \in R$.
Clearly d satisfies the triangle inequality d_5 and is not symmetric further
$\lim d(1 + \frac{1}{n}, 0) = \lim(1 + \frac{1}{n}) = 1$ and $\lim(0, 1 + \frac{1}{n}) = 0$

4 Continuity

Having defined two topologies on a distance space (X, d) , it is time now to study how the notion of continuity could be described in terms of the distances involved. At the very outset, we would like to point out how nonvanishing of self distances causes insurmountable hurdles in this venture. If (X, d_1) and (Y, d_2) are distance spaces with right topologies $\Im^r_{d_1}, \Im^r_{d_2}$ and $f : X \to Y$ is a constant mapping with $f(x) = y_0$ for all $x \in X$ where y_0 is an element of Y satisfying $d_2(y_0, y_0) > 0$ then f is obviously continuous. But if continuity is interpreted as a property of f that takes close-by points of X in to close-by points in Y then the same f fails to be continuous due to the nonvanishing of $d_2(y_0, y_0) > 0$. Hence nonvanishing of self distances creates a hiatus between topological notions and distance notions. Our usual notion that a metric gives more structure to the space than the topology induced by it, is being challenged here due to the presence of points with $d(x, x) > 0$.To obviate this difficulty, we impose the condition that $d(x, x) = 0; \forall x \in X$ on all distance spaces (X, d) in the rest of this paper. We feel that the above observations proclaim the wisdom of Frechet[1] in imposing the condition $d(x, x) = 0; \forall x \in X$ in a metric space.

Definition 4.1. Suppose $f : X \to Y$ where (X, d_1) and (Y, d_2) are distance spaces in which self distances are zero. Suppose $a \in X$ .We say that f is right continuous at a if to each $\epsilon > 0$ there corresponds a $\delta = \delta(\epsilon) > 0$, such that $d(fx, fa) < \epsilon$ whenever $d(x, a) < \delta$. We say that f is left continuous at a if to each $\epsilon > 0$ there corresponds a $\delta = \delta(\epsilon) > 0$ such that $d(fa, fx) < \epsilon$ whenever $d(a, x) < \delta$.

Remark 4.2. When $d(x, x) = 0 \forall x \in X, V^r_\epsilon(x) = B^r_\epsilon(x)$ and $V^l_\epsilon(x) = B^l_\epsilon(x)$ and for all $x \in X$ and all $\epsilon > 0$.

Theorem 4.3 (Local Continuity)**.** *Let (X, d_1) and (Y, d_2) be distance spaces in which self distances vanish. Let $a \in X$ and $f : X \to Y$. Then the following are equivalent:*

1. *f is right continuous at a*
2. *$x_n \xrightarrow{r} a$ in (X, d_1) implies $f(x_n) \xrightarrow{r} f(a)$ in (Y, d_2).*

Proof. Assume (a). Let $\epsilon > 0$. Choose $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ for all x satisfying $d(x, a) < \delta$. Since $x_n \xrightarrow{r} a$, there exist n_0 such that $d(x_n, a) < \delta$ for all $n \geq n_0$. Hence $d(f(x_n, f(a))) < \epsilon$ if $n \geq n_0$. Hence $f(x_n) \xrightarrow{r} f(a)$.
Assume(b). Suppose (a) is false. Then there exists $\epsilon > 0$ such that for each positive integer n there exists x_n in X such that $d(x_n, a) < \frac{1}{n}$ and $d(f(x_n, f(a))) \geq \epsilon$. Hence $x_n \xrightarrow{r} a$ in (X, d_1) but $f(x_n) \xrightarrow{r} f(a)$ in (Y, d_2). □

Theorem 4.4 (Global continuity)**.** *Let (X, d_1) and (Y, d_2) are distance spaces, $f : ((X, d_1)) \to (Y, d_2)$ is right continuous iff $f^{-1}(G)$ is right open in Y whenever G is right open in Y.*

Proof. Suppose f is right continuous. Let G be right open in Y and let $x \in f^{-1}(G)$. Since G is open and $f(x) \in G$, there exists a basic open set $V^r_\epsilon(f(x))$ such that $f(x) \in V^r_\epsilon(f(x)) \subseteq G$. Since f is right continuous, there exists $\delta > 0$ such that $V^r_\delta(f(x)) \subseteq V^r_\epsilon(f(x)) \subseteq G$.
Hence $V^r_\delta(f(x)) \subseteq f^{-1}(G)$, showing that $f^{-1}(G)$ is right open in X. Suppose, conversely, that $f^{-1}(G)$ is right open in X whenever G is right open in Y. Let $\epsilon > 0$ and let $a \in X$. Consider $G = V^r_\delta(f(a))$ which is obviously open. $f^{-1}(G)$ is open in X and $a \in V^r_\delta(f(x))$.
Hence there exists $\delta > 0$ such that $V^r_\delta(a) \subseteq f^{-1}(G)$.
$\Rightarrow f(V^r_\delta(a)) \subseteq G = V^r_\epsilon(f(a))$
$\Rightarrow x \in V^r_\delta(a) \Rightarrow f(x) \in f(V^r_\delta(a)) \subseteq V^r_\epsilon(f(a))$
$\Rightarrow d_2(f(a), f(x)) < \epsilon$.
This proves that f is continuous at every point of X. □

Definition 4.5. Let (X, d) be a distance space. If there is a real number $0 < c < 1$ such that $d(f(x), f(y)) < cd(x, y); \forall x, y$. Then f is called a contraction.

Definition 4.6. Let (X, d) be a distance space. And f is a contraction then it is right(left) continuous.

Proof. Suppose $x_n \xrightarrow{r} a$,
$i.e.$ $\lim d(x_n, x) = 0$.
Given $\epsilon > 0$ there exists $\delta > 0$ such that $d(x_n, x) < \frac{\epsilon}{c}$.
$\Rightarrow d(f(x_n), f(x)) < cd(x_n, x) < \epsilon$
$\Rightarrow d(f(x_n), f(x)) = 0$
$\Rightarrow f(x_n) \xrightarrow{r} f(x)$.
Similarly we can prove for left continuity. □

Definition 4.7. A sequence $\{x_n\}$ in a distance space (X, d) is a Cauchy sequence if for every $\epsilon > 0$ there corresponds a positive integer N such that $d(x_n, x_m) < \epsilon$ whenever $n \geq N$ and $m \geq N$.

Definition 4.8. A distance space (X, d) is said to be right(left) complete if every Cauchy sequence is right(left) convergent.

Theorem 4.9. *Let (X, d) be a right(left) complete distance space satisfying d_3, d_5 and f is a contraction then f has a unique fixed point.*

Proof. Let x, y be any two points of X. Since f is a contraction on X there exists a real number α with $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ then by induction, is any +ve integer n,
$d(f^n(x), f^n(y)) \leq \alpha^n d(x, y); \forall x, y \in X$.
Now let x_0 be any point of X.
Set for $n \leq 0, x_{n+1} = f(x_n = f^{n+1}(x_0))$.
Let $m, n (m > n)$ be any +ve integer. Then we have,
$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m)$
$= d(f^n(x_0), f^n(x_1)) + ... + d(f^{m-1}(x_0), f^{m-1}(x_1))$
$\leq [\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1}]d(x_0, x_1)$
$= \alpha^n[1 + \alpha + ... + \alpha^{m-n-1}]d(x_0, x_1)$
$< \frac{\alpha^n}{1-\alpha} d(x_0, x_1)$
Since $0 < \alpha < 1, \{x_n\}$ is a Cauchy sequence in (X, d).
Hence right convergent.
$i.e.\lim d(x_n, x) \to 0$
$\Rightarrow \lim d(f^n(x), x) = 0........(i)$
By the continuity of f,$\lim d(f(x_n), f(x)) \to 0$
$i.e.\lim d(f(x_{n+1}), f(x)) \to 0........(ii)$
From (i) and (ii), $f(x) = x$.
Uniqueness: $d(x, z) = d(fx, fz) < \alpha d(x, z) \Rightarrow d(x, z) = 0 \Rightarrow x = z$. □

Result 4.10. *Define $D(x, y) = d(x, y) + d(y, x)$ as in Theorem 2.16 then*

1. *$\lim D(x_n, x) = 0 \Leftrightarrow \lim d(x_n, x) = 0$ and $\lim d(x, x_n) = 0$.*
2. *X is D complete $\Leftrightarrow X$ is both right and left complete with respect to d.*

Proof. (1) is clear. Now we will prove (2).
Assume that X is both right and left complete with respect to d.
Let $\{x_n\}$ be a D Cauchy sequence in X and $\epsilon > 0$.
Then there exists a positive integer n_0 such that $m, n \geq n_0 \Rightarrow D(x_n, x_m) < \epsilon$
$\Rightarrow d(x_n, x_m) + d(x_m, x_n) < \epsilon$
$\Rightarrow d(x_n, x_m) < \epsilon$
Hence $\{x_n\}$ is a Cauchy's sequence with respect to d in X.
Since X is right and left complete with respect to d then there exists x in X such that $\lim d(x_n, x) = 0$ and $\lim d(x, x_n) = 0 \Rightarrow \lim D(x_n, x) = 0$.
Hence X is D Complete.
Conversely suppose that X is D complete and let $\{x_n\}$ be a Cauchy's sequence in X with respect to d.
Then for given $\epsilon > 0$, there exists a positive integer n_0 such that $m, n \geq n_0$
$\Rightarrow d(x_m, x_n) < \frac{\epsilon}{2}$
$\Rightarrow d(x_n, x_m) < \epsilon$

Since X is D complete, $\lim D(x_n, x) = 0$
$\Rightarrow \lim[d(x_n, x) + d(x, x_n)] = 0$
$\Rightarrow \lim d(x_n, x) = 0 = \lim d(x, x_n) = 0$
Hence X is right and left complete with respect to d.
Unless specified otherwise in what follows (X, D) stands for a D-space. □

Remark 4.11. D satisfies d_2 and d_5.

In the view of Result 4.10, we can derive a fixed point theorems for the distance space with respect to d if we can prove the same for D-space and derive the contractive inequality for D from d.

The D-space induced by a distance space with respect to d is very useful in deriving fixed point theorems for self-maps on a distance space (X, d) from their analogues for (X, D). If a self-map f on a distance space (X, d) satisfies a contractive inequality $d(f(x), f(y)) \leq \mathbb{S}_d(x, y)$, where $\mathbb{S}_d$ is a linear function of $\{d(u, v)/\{u, v\} \subseteq \{x, y, f(x), f(y)\}\}$ then f satisfies the contractive inequality $D(f(x), f(y)) \leq \mathbb{S}_D(x, y)$ where $\mathbb{S}_D$ is obtained by replacing d in $\mathbb{S}_d$ by D.

5 Conclusion

In this article we discuss topological properties of distance space and we give some pertinent results. Moreover, we give a few supporting examples to our results.

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