Advances in Analysis, Vol. 2, No. 2, April 2017
https://dx.doi.org/10.22606/aan.2017.22007

Order Theoretic Common n-tuple Fixed Point

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Abstract In this article, we solve an open problem initially suggested in [2], namely:

Let (X,d) be a Hausdorff left K-complete T_0-quasi-pseudometric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the preorder induced by ϕ. Let $F : X \times X \to X$ and $G_i : X \to X; i = 1, 2, \cdots, N$ for $N > 2$ be $N + 1$ d-sequentially continuous mapping on X such that the pairs $\{F; G_i\}; i = 1, 2, \cdots, N$ are weakly left-related.
Problem:

1. Can we prove that $F, G_1, \cdots, G_N$ have a common coupled fixed point in X?
2. Alternatively, what could be a correct formulation of the statement, using the induced preorder and the weakly left-related property that guarantees a positive answer?

We answer this question by the affirmative. In fact we prove that a more general result holds when $F : X^n \to X$ for a natural number $n > 2$.

Keywords: Quasi-pseudometric space, left K-complete, preordered space, left-weakly related, common couple fixed point, common n-tuple fixed

1 Preliminaries

Definition 1.1. *Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two prosets. A map $T : X \to Y$ is said to be **preorder-preserving** or **isotone** if for any $x, y \in X$,*

$$x \preceq_X y \Longrightarrow Tx \preceq_Y Ty.$$

*Similarly, for any family $(X_i, \preceq_{X_i}),\ i = 1, 2, \cdots, n;\ (Y, \preceq_Y)$ of posets, a mapping $F : X_1 \times X_2 \times \cdots \times X_n \to Y$ is said to be **preorder-preserving** or **isotone** if for any for any $(x_1, x_2, \cdots, x_n), (z_1, z_2, \cdots, z_n) \in X_1 \times X_2 \times \cdots \times X_n$,*

$$x_i \preceq_{X_i} z_i \text{ for all } i = 1, 2, \cdots, n \Longrightarrow F(x_1, x_2, \cdots, x_n) \preceq_Y F(z_1, z_2, \cdots, z_n).$$

Definition 1.2. *(Compare [4]) Let X be a non empty set. A function $d : X \times X \to [0, \infty)$ is called a **quasi-pseudometric** on X if:*

i) $d(x, x) = 0 \quad \forall\ x \in X,$
ii) $d(x, z) \le d(x, y) + d(y, z) \quad \forall\ x, y, z \in X.$

*Moreover, if $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$, then d is said to be a T_0**-quasi-pseudometric**. The latter condition is referred to as the T_0-condition.*

Remark 1.3.

– *Let d be a quasi-pseudometric on X, then the map d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X, called the **conjugate** of d. In the literature, d^{-1} is also denoted d^t or $\bar{d}$.*
– *It is easy to verify that the function d^s defined by $d^s := d \vee d^{-1}$, i.e. $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ defines a metric on X whenever d is a T_0-quasi-pseudometric on X.*

Definition 1.4. *Let (X,d) be a quasi-pseudometric space. The convergence of a sequence (x_n) to x with respect to $\tau(d)$, called **d-convergence** or **left-convergence** and denoted by $x_n \xrightarrow{d} x$, is defined in the following way*

$$x_n \xrightarrow{d} x \Longleftrightarrow d(x,x_n) \longrightarrow 0. \tag{1}$$

*Finally, in a quasi-pseudometric space (X,d), we shall say that a sequence (x_n) **d^s-converges** to x if it is both left and right convergent to x, and we denote it as $x_n \xrightarrow{d^s} x$ or $x_n \longrightarrow x$ when there is no confusion. Hence*

$$x_n \xrightarrow{d^s} x \iff x_n \xrightarrow{d} x \quad \text{and} \quad x_n \xrightarrow{d^{-1}} x.$$

Definition 1.5. *A sequence (x_n) in a quasi-pseudometric (X,d) is called*

*(a) **left d-Cauchy** if for every $\epsilon > 0$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that*

$$\forall\ n \geq n_0, \quad d(x,x_n) < \epsilon;$$

*(b) **left K-Cauchy** if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n,k : n_0 \leq k \leq n, \quad d(x_k,x_n) < \epsilon;$$

*(c) **d^s-Cauchy** if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall n,k \geq n_0, \quad d(x_n,x_k) < \epsilon.$$

Definition 1.6. *(Compare [4]) A quasi-pseudometric space (X,d) is called*

- ***left-K-complete** provided that any left K-Cauchy sequence is d-convergent,*
- **left Smyth sequentially complete** *if any left K-Cauchy sequence is d^s-convergent.*

Definition 1.7. *A T_0-quasi-pseudometric space (X,d) is called **bicomplete** provided that the metric d^s on X is complete.*

Definition 1.8. *Let (X,d) be a quasi-pseudometric type space. A function $T : X \to X$ is called **d-sequentially continuous** or **left-sequentially continuous** if for any d-convergent sequence (x_n) with $x_n \xrightarrow{d} x$, the sequence (Tx_n) d-converges to Tx, i.e. $Tx_n \xrightarrow{d} Tx$.*
*Similarly, a function $T : X_1 \times X_2 \times \cdots \times X_n \to X$ for $n \geq 2$, is said to be **d-sequentially continuous** or **left-sequentially continuous** if for any sequences (x_l^i) such that $x_l^i \xrightarrow{d} x^{*,i}$, then*

$$T(x_l^i, x_l^{i+1}, \cdots, x_l^n, x_l^i, \cdots, x_l^{i-1}) \xrightarrow{d} T(x^{*,i}, x^{*,i+1}, \cdots, x^{*,n}, x^{*,1}, \cdots, x^{*,i-1}).$$

Definition 1.9. *(Compare [1]) An element $(x^1, x^2, \cdots, x^n) \in X^n$ is called:*

*(E1) a **n-tuple fixed point** of the mapping $F : X^n \to X$ if*

$$F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) = x^i, \text{ for all } i, 1 \leq i \leq n.$$

*(E2) a **n-tuple coincidence point** of the mappings $F : X^n \to X$ and $T : X \to X$ if*

$$F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) = Tx^i$$

*for all $i, 1 \leq i \leq n$ and in this case $(Tx^1, Tx^2, \cdots, Tx^n)$ is called the **n-tuple point of coincidence**;*
*(E3) a **common n-tuple fixed point** of the mappings $F : X^n \to X$ and $T : X \to X$ if*

$$F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) = Tx^i = x^i$$

for all $i, 1 \leq i \leq n$.

From the above, we then obtained the natural following definitions

Definition 1.10. *Let X be a non empty set. An element $(x^1, x^2, \cdots, x^n) \in X^n$ is called:*

*(E'2) a **n-tuple coincidence point** of the mappings $F : X^n \to X$ and $G_k : X \to X$ with $k, 1 \leq k \leq N,\ N > 2$ if $F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) = G_k x^i$ for all $i, 1 \leq i \leq n$ and for all $k, 1 \leq k \leq N$;*

*(E'2) a **n-tuple common fixed point** of the mappings $F : X^n \to X$ and $G_1, G_2, \cdots, G_N : X \to X$ with $N > 2$ if $F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) = G_k x^i = x^i$ for all $i, 1 \leq i \leq n$ and for all $k, 1 \leq k \leq N$.*

Definition 1.11. *(See [2]) Let $(X, \preceq)$ be a preordered space, and $F : X^n \to X$ and $g : X \to X$ be two mappings. Then the pair $\{F, g\}$ is said to be **weakly left-related** if the following condition is satisfied:*
$(C1)$ $\quad F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) \preceq gF(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1})$
and

$$gx^i \preceq F(gx^i, gx^{i+1}, \cdots, gx^n, gx^1, \cdots, x^{i-1})$$

for all $1 \leq i \leq n$.

2 Main Result

We recall the following lemma.

Lemma 2.1. *(Compare [2]) Let (X, d) be a quasi-pseudometric space and $\phi : X \to \mathbb{R}$ a map. Define the binary relation " $\preceq$ " on X as follows:*

$$x \preceq y \Longleftrightarrow d(x, y) \leq \phi(y) - \phi(x).$$

Then " $\preceq$ " is a preorder on X. It will be called the preorder induced by ϕ.

Example 2.2. *Let $X = [0, \infty)$ and $d(x, y) = \max\{0, x - y\}$. Then (X, d) is a quasi-pseudometric space. For any positive real number t, let $\phi_t : X \to \mathbb{R}$ defined by $\phi_t(x) = tx$. Then for $x, y \in X$, we have*

$$\begin{aligned} x \preceq y &\Longleftrightarrow d(x, y) \leq \phi_t(y) - \phi_t(x) \\ &\Longleftrightarrow 0 = \max\{0, x - y\} \leq t(y - x). \end{aligned}$$

It follows that $3/2 \preceq 3,\ 1/2 \preceq 1$, etc.

Remark 2.3. *(Compare [2]) If in addition, the space (X, d) is T_0, then the relation $\overline{\preceq}$ defined by*

$$x \mathrel{\overline{\preceq}} y \Longleftrightarrow d^s(x, y) \leq \phi(y) - \phi(x).$$

is a partial order on X.

We introduce the following definition.

Definition 2.4. *Let $(X, \preceq)$ be a preordered set and $g, f : X \to X$. We say that the pair $\{g, f\}$ (in this order) is an **embedded pair** if*

$$g(x) \preceq f(g(x)),\ \text{whenever}\ \ x \in X.$$

*We shall say that the family $\{G_1, G_2, \cdots, G_n\}$ (in this order) is a n-**embedded chain** if for all $i = 1, \cdots, n-1$, the pair $\{G_i, G_{i+1}\}$ is an embedded pair. Observe that an embedded pair is a 2-embedded chain.*

*We shall say that the family $\{G_1, G_2, \cdots, G_n\}$ is a **dual** n-**embedded chain** if $\{G_1, G_2, \cdots, G_n\}$ and $\{G_n, G_{n-1}, \cdots, G_1\}$ are n-embedded chains.*

Example 2.5. *Let $X = [2, \pi)$ with the usual order and consider the pairs $\mathcal{F} = \{F_1(x) = 3x, F_2(x) = 5x\}$ and $\mathcal{G} = \{G_1(x) = \sin x + 1, G_2(x) = x^2\}$.*

For any $x \in X$,

$$F_1(x) = 3x \leq 5(3x) = F_2(F_1(x)) \text{ and } F_2(x) = 5x \leq 3(5x) = F_1(F_2(x)),$$

showing that $\mathcal{F}$ is a dual 2-embedded chain.

On the other way around

$$x \in X, G_1(x) = \sin x + 1 \leq (\sin x + 1)^2 = G_2(G_1(x)),$$

showing that $\mathcal{G}$ is an embedded pair, while

$$G_2(x) = x^2 > \sin(x^2) + 1 = G_1(G_2(x)),$$

showing that $\mathcal{G}$ is not a dual 2-embedded chain.

Now we are ready to give the solution to the open problem.

Theorem 2.6. *Let (Y,d) be a Hausdorff left K-complete T_0-quasi-pseudometric space, $\phi : Y \to \mathbb{R}$ be a bounded from above function and $\preceq$ the preorder induced by ϕ. Let $F : Y \times Y \to Y$ and $G_i : Y \to Y; i = 2, \cdots, r$ for $r > 2$ be $(r-1)+1$ d-sequentially continuous mapping on X such that the pairs $\{F, G_r\}; r = 2, 3$ are weakly left-related. Moreover, we assume that $\{G_r, G_{r-1}, \cdots, G_3\}$ is an $r-2$-embedded chain. Then $F, G_2, \cdots, G_r$ have a common n-tuple fixed point in Y.*

Proof. Let $X_0^1, \cdots, X_0^n \in X$. We construct the sequences $(X_l^i)_l$ in Y as follows:

$$G_r X^i_{rl-r} = X^i_{rl-r+1}, \cdots, G_3 X^i_{rl-3} = X^i_{rl-2},\ G_2 X^i_{rl-1} = X^i_{rl}$$

and

$$X^i_{rl-1} = F(X^i_{rl-2}, X^{i+1}_{rl-2}, \cdots, X^n_{rl-2}, X^1_{rl-2}, \cdots, X^{i-1}_{rl-2}),$$

for all $l \geq 1$. We shall show that

$$X_l^i \preceq X_{l+1}^i \quad \text{for all } i,\ 1 \leq i \leq n. \tag{2}$$

Since the pair $\{G_r, G_{r-1}\}$ is an embedded pair, we have

$$X_1^i = G_r X_0^i \preceq G_{r-1}(G_r X_0^i) = G_{r-1}(X_1^i) = X_2^i.$$

Again, since the pair $\{G_{r-1}, G_{r-2}\}$ is an embedded pair, we have

$$X_2^i = G_{r-1} X_1^i \preceq G_{r-2}(G_{r-1} X_1^i) = G_{r-2}(X_2^i) = X_3^i.$$

So we obtain recursively

$$G_r X_0^i = X_1^i \preceq G_{r-1} X_1^i = X_2^i \preceq \cdots \preceq G_3(X^i_{r-3}) = X^i_{r-2}.$$

Now, since the pair $\{F, G_3\}$ is weakly left-related, we have

$$\begin{aligned} X^i_{r-2} = G_3(X^i_{r-3}) &\preceq F(G_3 X^i_{r-3}, G_3 X^{i+1}_{r-3}, \cdots, G_3 X^n_{r-3}, G_3 X^1_{r-3}, \cdots, G_3 X^{i-1}_{r-3}) \\ &= F(X^i_{r-2}, X^{i+1}_{r-2}, \cdots, X^n_{r-2}, X^1_{r-2}, \cdots, X^{i-1}_{r-2}) = X^i_{r-1}. \end{aligned}$$

Again since the pair $\{F, G_2\}$ is weakly left-related, we have

$$\begin{aligned} X^i_{r-1} &= F(X^i_{r-2}, X^{i+1}_{r-2}, \cdots, X^n_{r-2}, X^1_{r-2}, \cdots, X^{i-1}_{r-2}) \\ &\preceq G_2 F(X^i_{r-2}, X^{i+1}_{r-2}, \cdots, X^n_{r-2}, X^1_{r-2}, \cdots, X^{i-1}_{r-2}) \\ &= G_2 X^i_{r-1} = X^i_r. \end{aligned}$$

Similarly, using repeatedly the fact that the pairs $\{F, G_2\}$ and $\{F, G_3\}$ are weakly left-related, and that $\{G_r, G_{r-1}, \cdots, G_3\}$ is an an $r-2$-embedded chain, we obtain

$$X_1^i \preceq X_2^i \preceq X_3^i \preceq \ldots \preceq X_l^i \preceq \cdots .$$

By definition of the preorder, we have

$$\phi(X_1^i) \leq \phi(X_2^i) \leq \ldots \leq \phi(X_l^i) \leq \cdots ,$$

Hence, the sequence $(\phi(X_l^i))$ is a non-decreasing sequence of real numbers. Since ϕ is bounded from above, the sequence $(\phi(X_l^i))$ converges and is therefore Cauchy. This entails that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $q > p > n_0$, we have $\phi(X_q^i) - \phi(X_p^i) < \varepsilon$. Since whenever $q > p > n_0$, $X_p^i \preceq X_q^i$ and it follows that

$$d(X_p^i, X_q^i) \leq \phi(X_q^i) - \phi(X_p^i) < \varepsilon.$$

We conclude that (X_l^i) is a left K-Cauchy sequence in Y and since Y is left K-complete, there exist $X^{*,i} \in Y$ such that $X_l^i \xrightarrow{d} X^{*,i}$.
Since F and $G_2, \cdots, G_r$ are d-sequentially continuous, it is easy to see that

$$\begin{aligned} X_{rl-1}^i \xrightarrow{d} X^{*,i} &\Longleftrightarrow F(X_{rl-2}^i, X_{rl-2}^{i+1}, \cdots, X_{rl-2}^n, X_{rl-2}^1, \cdots, X_{rl-2}^{i-1}) \xrightarrow{d} X^{*,i} \\ &\Longleftrightarrow F(X^{*,i}, X^{*,i+1}, \cdots, X^{*,n}, X^{*,1} \cdots, X^{*,i-1}) = X^{*,i} \end{aligned}$$

and

$$X_{rl-r}^i \xrightarrow{d} X^{*,i} \Longleftrightarrow X_{rl-k+1}^i = G_k X_{rl-k}^i \xrightarrow{d} X^{*,i} \Longleftrightarrow G_k X^{*,i} = X^{*,i},$$

and hence

$$G_k X^{*,i} = X^{*,i} = F(X^{*,i}, X^{*,i+1}, \cdots, X^{*,n}, X^{*,1} \cdots, X^{*,i-1}).$$

Hence $(X^{*,1}, X^{*,2}, \cdots, X^{*,n})$ is a common n-tuple fixed point of F and $G_2, \cdots, G_r$.

□

Example 2.7. *Let $X = [0, \infty)$ and $d : X \times X \to \mathbb{R}$ be the mapping defined by $d(a, b) = \max\{a-b, 0\}$. Then d is a T_0-quasi-pseudometric on X. Observe that any left K-Cauchy sequence in (X, d) is d-convergent to 0. Indeed, if (x_n) is a left K-Cauchy sequence, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n, k : n_0 \leq k \leq n \quad d(x_k, x_n) < \epsilon.$$

This entails that $\forall\ n : n_0 < n$

$$d(0, x_n) \leq d(0, x_{n-1}) + d(x_{n-1}, x_n) = 0 + d(x_{n-1}, x_n) < \epsilon.$$

Hence $d(0, x_n) \longrightarrow 0$, i.e. $x_n \xrightarrow{d} 0$. Therefore (X, d) is left K-complete.
For any positive real number a, let $\phi_a : X \to \mathbb{R}$ be defined by $\phi_a(x) = ax$, and $\preceq$ the preorder induced by ϕ_a. We define $F : X^n \to X$ and $GX \to X$ as follows

$$F(x^1, x^2, \cdots, x^n) = x^1 + |\sin(x^1 x^2 \cdots x^n)| \text{ and } G_k(x) = kx, k = 2, \cdots, r,\ r > 2.$$

For $k = 1, 2$, we have on one hand,

$$G_k F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) = k(x^i + |\sin(x^1 x^2 \cdots x^n)|),$$

i.e.

$$F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}) \preceq G_k F(x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-1}),$$

and on the other hand,

$$F(G_kx^i, G_kx^{i+1}, \cdots, G_kx^n, G_kx^1, \cdots, G_kx^{i-1}) = F(kx^i, kx^{i+1}, \cdots, kx^n, kx^1, \cdots, kx^{i-1}) \\ = kx^i + |\sin(k^n x^1 x^2 \cdots x^n)|,$$

i.e.

$$G_kx^i \preceq F(G_kx^i, G_kx^{i+1}, \cdots, G_kx^n, G_kx^1, \cdots, G_kx^{i-1}).$$

And so the pair $\{F, G_k\}$ *are weakly left-related for* $k = 2, 3$. *Again, it is not hard to see that* F *and* $G_k, k = 2, \cdots, r$, *are d-sequentially continuous mappings on* X.
Moreover, for any $x \in [0, \infty)$, $kx \leq k(k-1)x, k = 2, \cdots, r$, *implying that* $\{G_r, G_{r-1}, \cdots, G_3\}$ *is an an* $r - 2$*-embedded chain. Hence we see that all the conditions of our theorem are satisfied.*
Also we have

$$F(0, x^i, x^{i+1}, \cdots, x^n, x^1, \cdots, x^{i-2}) = 0 = G_k(0)$$

for $k = 2, \cdots, r$ *and for* $i = 1, 2, \cdots, n$.
Thus $\underbrace{(0, \cdots, 0)}_{n}$ *is a common n-tuple fixed point of* $F, G_2, \cdots, G_r$.

Corollary 2.8. *Let (Y,d) be a Hausdorff left K-complete* T_0*-quasi-pseudometric space,* $\phi : Y \to \mathbb{R}$ *be a bounded from above function and* $\preceq$ *the preorder induced by* ϕ. *Let* $F : Y \times Y \to X$ *and* $G_i : Y \to Y; i = 1, 2, \cdots, r$ *for* $r > 2$ *be* $N + 1$ *d-sequentially continuous mapping on* X *such that the pairs* $\{F, G_r\}; r = 2, 3$ *are weakly left-related. Moreover, we assume that* $\{G_r, G_{r-1}, \cdots, G_3\}$ *is a dual* $r - 2$*-embedded chain. Then* $F, G_2, \cdots, G_r$ *have a common n-tuple fixed point in* Y.

Corollary 2.9. *Let (Y,d) be a bicomplete* T_0*-quasi-pseudometric space,* $\phi : Y \to \mathbb{R}$ *be a bounded from above function and* $\overline{\preceq}$ *the preorder induced by* ϕ. *Let* $F : Y \times Y \to X$ *and* $G_i : Y \to Y; i = 1, 2, \cdots, r$ *for* $r > 2$ *be* $N + 1$ d^s*-sequentially continuous mapping on* X *such that the pairs* $\{F, G_r\}; r = 2, 3$ *are weakly left-related. Moreover, we assume that* $\{G_r, G_{r-1}, \cdots, G_3\}$ *is an* $r - 2$*-embedded chain. Then* $F, G_2, \cdots, G_r$ *have a common n-tuple fixed point in* Y.

3 Concluding Remark

All the results given remain true when we replace accordingly the bicomplete quasi-pseudometric space (X, d) by a left Smyth sequentially complete/left K-complete or a right Smyth sequentially complete/right K-complete space.

References

1. Ertürk, M, Karakaya, V; n-tuple fixed point theorems for contractive type mappings in partially ordered metric spaces. J. Inequal. Appl. 2013, 196 (2013).
2. Y. U. Gaba; "An order theoretic approach in fixed point theory." Mathematical Sciences (2014): 1-7. DOI 10.1007/s40096-014-0133-6.
3. V. Lakshmikantham and Lj. B. Ciric; Coupled fixed point theorems for nonlinear contractions in partially ordered metric space, Nonlinear Anal. 70, no. 12, 4341-4349, 2009.
4. Y. U. Gaba; Startpoints and (α, γ)-contractions in quasi-pseudometric spaces, Journal of Mathematics, Volume 2014 (2014), Article ID 709253, 8 pages http://dx.doi.org/10.1155/2014/709253.

