Generalized Taylor Series

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Abstract In [1] we defined a notion of a generalized derivative for functions defined on a general topological space with values in a linear topological space. Here we develop a theory of Taylor series for this generalized setting.

Keywords: Generalized derivatives, calculus, Taylor series.

1 Introduction

When working with mathematical objects and notions, we often use more structural properties than it is needed (see [2-7]). During last years notions similar to the classical notions of uniform convergence, periodicity, Lipschitz condition and others have been defined in a purely topological way. (See e. g. [8],[9], [10] or [11]).

Let us note that the generalized differentiation is currently developing by several mathematicians in the frame of the theory of arbitrary metric spaces. See, for example,[12],[13],[14],[15]. The differentiation theory in linear topological spaces is a well-known part of analysis. Nonetheless, it seems that there were not any attempts to introduce a differentiation in topological spaces without linear or metric structures before [1] was published.

2 Generalized Derivative

In this paper, when we say "a field", we mean the spaces R or C equipped with their natural topologies. But very often we could work with a general "topological field", a field on which the operations of addition, subtraction, multiplication and division would be continuous. In what follows we will suppose that X is a topological space, $A \subset X$ and Y is a linear topological space defined over a field F. A function g defined on X is discrete on A at a point $a \in X$, if there is an open neighborhood V of a such that the statement

$$g(a) \notin g((V \bigcap A) \setminus \{a\})$$

holds. It is clear that g is a discrete function (i. e. the set $g^{-1}(g(a))$ is discrete for every $a \in X$), if and only if g is discrete on X at all points. Now we define the notion of generalized derivative.

Definition 2.1. ([1]) Let p be a limit point of A and $g : X \to F$ be discrete at p on A. A function $f : X \to Y$ has a g-derivative $l \in Y$ at p on A if for every net $\{x_\gamma\}_{\gamma \in \Gamma}$ of points $x_\gamma \in A \setminus \{p\}$ converging to p, the net $\{\frac{f(x_\gamma)-f(p)}{g(x_\gamma)-g(p)}\}_{\gamma \in \Gamma}$ converges to l. If l is a g-derivative of f at p on A, then we write

$$l =_{g/A} f^{'}(p) = lim_{x \to p, x \in A} \frac{f(x)-f(p)}{g(x)-g(p)}$$

and ${}_g f^{'}(p) =_{g/X} f^{'}(a)$ for $A = X$.

It is easy to see that, for Hausdorff spaces Y, a g-derivative ${}_g f^{'}(p)$ if it exists, is unique (see [1]).

Remark 2.2. In what follows if we say "let ${}_{g/A} f^{'}(a)$ exist" we automatically suppose that a is a limit point of A. Moreover, we see, that if ${}_{g/A} f^{'}(a)$ exists, then g is discrete on A at the point a. It is easy to see that our generalized derivative is a linear operator. If we put $X = A = Y = F = R$ and $g(x) \equiv x$ we obtain the well-known definition of the derivative of a real valued function. Functions, differentiable in this generalized sense need not be continuous. For example if $g : R \to R$ is an injective noncontinous function and $f = g$ then for every a from R $\ _g f^{'}(a) = 1$ so $\ _g f^{'}(a)$ exists. But f is not continuous. It can be proved, though, that if g is continuous, f arbitrary and $\ _g f^{'}$ exists, then f is continuous.

The following example illustrates the notion of the generalized derivative.

Example 2.3. Let X be the space of all functions that are Riemann integrable on $< 0, 1 >$, let X be equipped with the topology of pointwise convergence. Be k a positive integer. We define two mappings, i and j from X to R by

$\forall f \in X \ i(f) = \int_0^1 f^k(x)dx$

$\forall f \in X \ j(f) = \int_0^1 f(x)dx$

For each $n = 1, 2, \ldots$ define $f_n :< 0, 1 >\to R$ by $\forall x \in< 0, 1 >$ $f_n(x) = x^n$. Define $h :< 0, 1 >\to R$ by $h(x) = 0$ on $< 0, 1)$ and $h(1) = 1$. Define $A \subset X$ by $A = \{f_n : n = 1, 2, \ldots\} \cup \{h\}$.

We can see that h is a limit point of A. Let us count $\ _j i^{'}(h)$.

$\ _j i^{'}(h) = lim_{f \in A, f \to h} \frac{i(f)-i(h)}{j(f)-j(h)} = lim_{n\to\infty} \frac{i(f_n)-i(h)}{j(f_n)-j(h)} = lim_{n\to\infty} \frac{\frac{1}{kn+1}-0}{\frac{1}{n+1}-0} = \frac{1}{k}$.

Now what is the "meaning" of the fact that $\ _j i^{'}(h) = \frac{1}{k}$? One interpretation of this fact is that as n approaches the infinity (and the functions f_n approach h), the quotient of $i(f_n)$ and $j(f_n)$ aproaches $\frac{1}{k}$ so for sufficiently high indexes n we obtain

$i(f_n) \approx \frac{1}{k} j(f_n)$ or more concretely $\int_0^1 x^{kn}(x)dx \approx \frac{1}{k} \int_0^1 x^n(x)dx$.

The following technical lemma will be needed for our theory of generalized Taylor series in linear topological spaces. It can also serve as another example illustrating generalized derivative.

Lemma 2.4. *Let (X, T) be a topological space. Let $A \subset X$ and let a be a limit point of A. Let $g : X \to R$ be a function continuous at a with respect to A. Let g be discrete on A at a. Be n a positive integer and $c \in R$. Put $h(x) := (g(x) - c)^n$. Then there exists $\ _{g/A}h'(a) = \ _{g/A}((g(x) - c)^n)'(a)$ and $\ _{g/A}h'(a) = n(g(a) - c)^{n-1}$.*

Proof. $\ _{g/A}((g(x) - c)^n)'(a) = lim_{x\in A, x\to a} \frac{(g(x)-c)^n-(g(a)-c)^n}{(g(x)-g(a))} =$

$lim_{x\in A, x\to a} \frac{((g(x)-c)-(g(a)-c))((g(x)-c)^{n-1}+(g(x)-c)^{n-2}(g(a)-c)+\cdots+(g(a)-c)^{n-1})}{(g(x)-g(a))} =$

$lim_{x\in A, x\to a} \frac{(g(x)-g(a))((g(x)-c)^{n-1}+(g(x)-c)^{n-2}(g(a)-c)+\cdots+(g(a)-c)^{n-1})}{(g(x)-g(a))} =$

$lim_{x\in A, x\to a}(g(x) - c)^{n-1} + (g(x) - c)^{n-2}(g(a) - c) + \cdots + (g(a) - c)^{n-1}) = n(g(a) - c)^{n-1}$.

3 Generalized Mean Value Theorems

A function $g : X \to R$ will be called *feebly monotone* at $p \in X$ on A, if for every open $O \ni p$ there exist $s, t \in O \bigcap A$, such that the inequalities

$$g(s) < g(p) < g(t)$$

hold.

The following lemma ([1]) is a generalization of the fact, that if a differentiable function f has an extremum at s then $f^{'}(s) = 0$.

Lemma 3.1.([1]) *Let s be a limit point of A and let $g : X \to R$ be feebly monotone and discrete at s on A. If a function $f : X \to R$ has at s a g/A-derivative and a local extremum on A, then*

(i) $\ _{g/A}f^{'}(s) = 0$

is true.

Now we give a generalization of the Rolle‘s Theorem.

Theorem 3.2. *Let (X, T) be a topological space, $K \subset X$ be a compact. Let S be a subset of $K$and let $A = K - S$ be nonempty and such, that each t from A is a limit point of A. Let $f : X \to R$ and $g : X \to R$ be functions. Let f be continuous on K, let f be constant on S. Let g be discrete on K (on A) and let it have no local extrema on K (on A) with respect to K (with respect to A). Let for all x from A there exist $\ _{g/K}f^{'}(x)$ ($\ _{g/A}f^{'}(x)$). Then there exists a point c from A such that $\ _{g/K}f^{'}(c) = 0$ ($\ _{g/A}f^{'}(c) = 0$).*

Proof. If f is constant on K our theorem is true. If it is not constant, the set $f(K)$ is compact in R and it has more than one point. So there exists a point c from A in which f has a local maximum or a local minimum with respect to K. Therefore ${}_{g/K}f^{'}(c) = 0$.

The "$({}_{g/A}f^{'}(c) = 0)$" part of the theorem can be proved similarly. If f is constant on A our theorem is true. If it is not constant, the set $f(K)$ is compact in R and has more than one point. So there exists a point c from A in which f has a local maximum or a local minimum with respect to K and with respect to A. Therefore ${}_{g/A}f^{'}(c) = 0$.

Observe that if S is the boundary of K then A is the interior of K and local extrema in A with respect to A are in fact local extrema with respect to X. So if ${}_{g}f^{'}$ exists on A, there exists a point c from A such that ${}_{g}f^{'}(c) = 0$. A classic example of this situation is $K =< a, b >, S = \{a, b\}, A = (a, b)$, $g(x)$ being the idendity function on R.

The following theorem is a generalization of the Mean Value Theorem.

Theorem 3.3. *Let (X, T) be a topological space. Let $K \subset X$ be a compact. Let $S = \{a, b\}$ be a subset of K and let $A = K - S$ be nonempty and such, that each t from A is a limit point of A. Let $f : K \to R$, $h : K \to R$ and $g : K \to R$ be functions. Let f and h be continuous on K. Let g be discrete on A (on K) and let it have no local extrema on A (on K) with respect to A (with respect to K). Let for all x from A there exist ${}_{g/A}f^{'}(x)$ and ${}_{g/A}h^{'}(x)$ (${}_{g/K}f^{'}(x)$ and ${}_{g/K}h^{'}(x)$). Then there exists a point c from A such that*

${}_{g/A}h^{'}(c)(f(b) - f(a)) = {}_{g/A}f^{'}(c)(h(b) - h(a))$

(or ${}_{g/K}h^{'}(c)(f(b) - f(a)) = {}_{g/K}f^{'}(c)(h(b) - h(a))$).

If c is from the interior of $K - \{a, b\}$ we obtain

${}_{g}h^{'}(c)(f(b) - f(a)) = {}_{g}f^{'}(c)(h(b) - h(a))$

Proof. Define a function $p(x)$ such that for all x from K

$p(x) = (h(x) - h(a))(f(b) - f(a)) - (f(x) - f(a))(h(b) - h(a))$.

The function p is continuous on K and $p(a) = p(b) = 0$. This means there exists a point c from $K - \{a, b\}$ such that p has a local maximum (or minimum) at c. So ${}_{g/A}p^{'}(c) = 0$ (or ${}_{g/K}p^{'}(c) = 0$). Moreover for all x from $K - \{a, b\}$ we have ${}_{g/A}p^{'}(x) = {}_{g/A}h^{'}(x)(f(b) - f(a)) - {}_{g/A}f^{'}(x)(h(b) - h(a))$. Concretely, this means ${}_{g/A}h^{'}(c)(f(b) - f(a)) = {}_{g/A}f^{'}(c)(h(b) - h(a))$ is true.

Of course, if c is from the interior of $K - \{a, b\}$ then we obtain

${}_{g/A}p^{'} = {}_{g/K}p^{'} = {}_{g}p^{'}$ so

${}_{g}h^{'}(c)(f(b) - f(a)) = {}_{g}f^{'}(c)(h(b) - h(a))$

Let us remark that since we work on a general set, maybe some other, more interesting type of function could be used for $p(x)$. In that case S could have more than two points that would be involved in our formula. The function we use is the traditional type of function, used for intervals.

The preceding theorem is a generalization of the so called Cauchy Mean Value Theorem. (Another kind of this theorem was proved in [1]. The result obtained here is different and we need it in order to develop our generalized theory of Taylor series.) Very often we need a simpler version of the proved formula. The following theorem is the consequence of the preceding one.

Theorem 3.4. *Let (X, T) be a topological space. Let $K \subset X$ be a compact. Let $S = \{a, b\}$ be a subset of K and let $A = K - S$ be nonempty and such, that each t from A is a limit point of A. Let $f : K \to R$, and $g : K \to R$ be functions. Let f and g be continuous on K, let f be constant on S. Let g be discrete on A and let it have no local extrema on A with respect to A. Let for all x from A there exist${}_{g/A}f^{'}(x)$. Then there exists a point c from A such that*

$(f(b) - f(a)) = {}_{g/A}f^{'}(c)(g(b) - g(a))$.

If c is from the interior of $K - \{a, b\}$ we obtain

$(f(b) - f(a)) = {}_{g}f^{'}(c)(g(b) - g(a))$

Proof. Define a function $h : K \to R$ such that for all x from K $h(x) = g(x)$. The assumptions of the preceding g theorem are met so there exists a point c from $A = K - \{a, b\}$ such that

${}_{g/A}h^{'}(c)(f(b) - f(a)) = {}_{g/A}f^{'}(c)(h(b) - h(a))$.

Since $g = h$ the function ${}_{g/A}h^{'}$ equals 1 on its whole range. Realizing this and replacing h by g on the right side we obtain

$(f(b) - f(a)) = {}_{g/A}f^{'}(c)(g(b) - g(a))$.

Example 3.5. Our new mean value theorem is more general than the classical one. This example shows that on one hand, we should not be overenthusiastic about it, on the other hand, we really can obtain better estimates than in the classical case.

(1)

Put $f(x) = x^2$ on $K =< 0, \frac{1}{3} > \cup < \frac{2}{3}, 1 >$ and $g(x) = x = h(x)$ on K. Since $f(1) - f(0) = 1$ there should exist a point c from $A = (0, \frac{1}{3} > \cup < \frac{2}{3}, 1)$ such that

${}_{g/A}h^{'}(c)(f(1) - f(0)) = {}_{g/A}f^{'}(c)(h(1) - h(0))$.

Since in our case ${}_{g/A}f^{'}(c) = {}_{x}f^{'}(c) = f^{'}(c) = 2c$ and ${}_{g/A}h^{'}(c) = {}_{h}h^{'}(c) = 1$ we should obtain

(*) there exists c from A such that $1 = 2c$

if our generalized Mean Value Theorem worked. But obviously c cannot be an element of

$(0, \frac{1}{3} > \cup < \frac{2}{3}, 1)$. At the first sight all the conditions of our theorem are met. But examining the case better we can see, that g does not satisfy the condition "*let g have no local extrema on A*".

Indeed g has local extrema at the points $\frac{1}{3}$ and $\frac{2}{3}$.

(2)

Define $f(x) = x^8$ on $K =< \frac{1}{4}, \frac{1}{2} >$.

Without using any kind of Mean Value Theorem we can see that $|f(\frac{1}{2}) - f(\frac{1}{4})| = (\frac{1}{2})^8 - (\frac{1}{4})^8 \leq (\frac{1}{2})^8$

Let us use the classical Mean Value Theorem to estimate this difference.

There exists c from $(\frac{1}{4}, \frac{1}{2})$ such that $|f(\frac{1}{2}) - f(\frac{1}{4})| = f^{'}(c)(\frac{1}{2} - \frac{1}{4}) = 8c^7 \cdot \frac{1}{4}$. Since c could be very near to the point $\frac{1}{2}$ we obtain only

$|f(\frac{1}{2}) - f(\frac{1}{4})| < 8 \cdot (\frac{1}{2})^7 \cdot \frac{1}{4} = \frac{1}{2^6}$

Now use the preceding theorem on $< \frac{1}{4}, \frac{1}{2} >$ for functions $f(x) = x^8$ and $g(x) = x^4$. First count ${}_{g}f^{'}(x)$ in general.

${}_{g}f^{'}(x) = lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)} = lim_{t \to x} \frac{t^8 - x^8}{t^4 - x^4} = 2x^4$. According our theorem there exists c from $(\frac{1}{4}, \frac{1}{2})$ such that

$(f(\frac{1}{2}) - f(\frac{1}{4})) = {}_{g}f^{'}(c)(g(\frac{1}{2}) - g(\frac{1}{4})) = 2c^4 \cdot ((\frac{1}{2})^4) - (\frac{1}{4})^4)$. Since c could be very near to the point $\frac{1}{2}$ we obtain $|f(\frac{1}{2}) - f(\frac{1}{4})| < 2 \cdot (\frac{1}{2})^4 \cdot (\frac{1}{2^4} - \frac{1}{4^4}) < \frac{1}{2^3} \cdot \frac{1}{2^4} = \frac{1}{2^7}$. This estimate is better than the classical one.

4 Generalized Taylor Series

Theorem 4.1. *Let (X, T) be a topological space, Y a linear topological space over F (where $F = R$ or $F = C$). Let $f : X \to Y$ and $g : X \to F$ be functions. Let a be a limit point in X. Let g be continuous and discrete on an open neighborhood of a. Let n be a positive integer and let for every $k = 1, 2, \ldots n$ there exist ${}_{g}f^{(k)}(a)$ (this implies there exists an open neighborhood U of a such that for each x from U and for every $k = 1, 2, \ldots n - 1$ ${}_{g}f^{(k)}(x)$ exists).*

Then there exists exactly one n-tuple of vectors $A_0, A_1, \ldots A_n$ from Y such that the function $T_n : X \to Y$ defined by

$\forall x \in X \quad T_n(x) = A_0 + A_1(g(x) - g(a)) + \cdots + A_n(g(x) - g(a))^n$

fulfills

$T_n(a) = f(a)$

${}_{g}T_n^{'}(a) = {}_{g}f^{'}(a)$

$\ldots$

${}_{g}T_n^{(n)}(a) = {}_{g}f^{(n)}(a)$

Moreover the following holds:

for every $k = 1, 2, \ldots n$ $A_k = \frac{{}_{g}f^{(k)}(a)}{k!}$

Proof. If we put $A_k = \frac{{}_{g}f^{(k)}(a)}{k!}$ for each $k = 1, 2, \ldots n$, the function $T_n(x)$ has the demanded properties.

On the other hand let $\forall x \in X \quad T_n(x) = A_0 + A_1(g(x) - g(a)) + \cdots + A_n(g(x) - g(a))^n$ and let $T_n(a) = f(a)$. We can see immediately that $T_n(a) = A_0$ so we have $A_0 = f(a)$.

In what follows we use the fact that if A is a vector from Y and $h : X \to R$ is a function and if ${}_{g}h^{'}(a)$ exists then the function $H : X \to Y$ defined by $H(x) = Ah(x)$ has a g-derivative at a and ${}_{g}(Ah)^{'}(a) = A_{g}h^{'}(a)$.

Using Lemma 2.4. we obtain ${}_gT_n^{'}(x) = \sum_{k=0}^{n} {}_g(A_k(g(x)-g(a))^k)^{'} = \sum_{k=0}^{n}(A_k{}_g((g(x)-g(a))^k)^{'} = \sum_{k=1}^{n}(A_k k(g(x)-g(a))^{k-1})$ for each x at which these derivatives exist. This implies ${}_gT_n^{'}(a) = A_1$ so $A_1 = {}_gf^{'}(a)$.

In general if $1 \leq j \leq n$ then

${}_gT_n^{(j)}(x) = \sum_{k=j}^{n}(A_k k(k-1)\dots(k-j+1)(g(x)-g(a))^{k-j})$. This means ${}_gT_n^{(j)}(a) = j!A_j$. Since ${}_gT_n^{(j)}(a) = {}_gf^{(j)}(a)$ we have $A_j = \frac{{}_gf^{(j)}(a)}{j!}$. This ends the proof.

Now we show how to estimate the difference between the generalized Taylor polynomial of degree n and the original function f. We provide here a generalized Cauchy form of the reminder term.

Theorem 4.2. *Let (X,T) be a topological space. Let K be a compact subset of X that has no isolated points with respect to the relative topology on K. Let $S = \{a,x\}$ be a subset of $K$and let $A = K - S$ be nonempty. Let X be T_1 (this assures each t from A is a limit point of A).*

Let $f : X \to R$, and $g : X \to R$ be functions. Let f and g be continuous on K. Let g be discrete on A and let it have no local extrema on A with respect to A. Be n a positive integer. Let for all t from A there exist ${}_{g/A}f^{n+1}(t)$ and let ${}_{g/K}f^{j}$ exist and be continuous on K for all $j = 1,2,\dots n$. Denote

$T_n(f,g,a,x) = f(a) + (g(x)-g(a))_{g/K}f^{'}(a) + \dots + \frac{(g(x)-g(a))^n}{n!}{}_{g/K}f^{(n)}(a)$

Then there exists an element c from $K - \{a,x\}$ such that

$f(x) - T_n(f,g,a,x) = (g(x)-g(a))\frac{(g(x)-g(c))^n}{n!}{}_{g/A}f^{(n+1)}(c)$

and if A is open

$f(x) - T_n(f,g,a,x) = (g(x)-g(a))\frac{(g(x)-g(c))^n}{n!}{}_{g}f^{(n+1)}(c)$

Proof. Put

$F(t) = f(t) + (g(x)-g(t))_{g/K}f^{'}(t) + \dots + \frac{(g(x)-g(t))^n}{n!}{}_{g/K}f^{(n)}(t)$

F is continuous on K

We can see that $F(x) - F(a) = f(x) - T_n(f,g,a,x)$

By the generalized Mean Value Theorem there exists an element c from $K - \{a,x\}$ such that

$F(x) - F(a) = (g(x)-g(a))_{g/A}F^{'}(c)$

So if we show

${}_{g/A}F^{'}(c) = \frac{(g(x)-g(c))^n}{n!}{}_{g/A}f^{(n+1)}(c)$

we are done.

But this is true since for each t from $K - \{a,b\}$ and for $j = 1,2,\dots n$ obviously ${}_{g/A}f^{j}(t) ={}_{g/K} f^{j}(t)$ and we have

${}_{g/A}F^{'}(t) = {}_{g/A}\{f(t) + (g(x)-g(t))_{g/A}f^{'}(t) + \frac{(g(x)-g(t))^2}{2!}{}_{g/A}f^{(2)}(t) + \dots + \frac{(g(x)-g(t))^n}{n!}{}_{g/A}f^{(n)}(t))\} = {}_{g/A}f^{'}(t) + (-1)_{g/A}f^{'}(t) + (g(x)-g(t))_{g/A}f^{(2)}(t) + (-1)2\frac{(g(x)-g(t))}{2!}{}_{g/A}f^{(2)}(t) + \frac{(g(x)-g(t))^2}{2!}{}_{g/A}f^{(3)}(t)) + \dots$

$+(-1)n\frac{(g(x)-g(t))^{n-1}}{n!}{}_{g/A}f^{(n)}(t)) + \frac{(g(x)-g(t))^n}{n!}{}_{g/A}f^{(n+1)}(t) =$

$\frac{(g(x)-g(t))^n}{n!}{}_{g/A}f^{(n+1)}(t)$.

This ends the proof.

References

1. I. Kupka, "Topological Generalization of Cauchy's Mean Value Theorem," *Ann. Accad. Sci. Fenn. Math. Diss.,* vol. 41, pp. 315–320, 2016.

2. K. Deimling, "Nonlinear Functional Analysis," *Springer-Verlag,* Berlin, Heidelberg 1985.

3. R. Engelking, "General Topology," *PWN Warszaw,* 1977.

4. M. Grossman and R. Katz, "Non-Newtonian Calculus," 7'th printing, *Mathco,* Rockport, Massachusetts, 1979.

5. G. L. Light, "An Introductory Note on Relative Derivative and Proportionality", *Int. J. Contemp. Math. Sci.,* vol. 1, no. 7, pp. 327–332, 2006.

6. A. P. Robertson and W. Robertson, "Topological vector spaces," *Cambridge University Press,* Cambridge, 1964.

7. V. Aversa, D. Preiss, "Lusins theorem for derivatives with respect to a continuous function," *Proc. Amer. Math. Soc.,* vol. 127, pp. 3229–3235, 1999.

8. I. Domnik, "On strong convergence of multivalued maps," *Math. Slovaca,* vol. 53, no. 2, pp. 199–209, 2003.
9. I. Kupka, "A Topological Analogue of the Lipschitz Condition," *Topology Proceedings,* vol. 32, pp. 75–81, 2008.
10. I. Kupka, "Invariance and Periodicity," *Demonstratio Mathematica,* vol. 40, no. 2, pp. 457–464, 2007.
11. V. Toma, "Strong convergence and Dini theorems for non-uniform spaces," *Ann. Math. Blaise Pascal,* vol. 4, no. 2, pp. 97–102, 1997.
12. L. Ambrosio and B. Kirchheim, "Rectifiable sets in metric and Banach spaces," *Math. Ann.,* vol. 318, pp. 527-555, 2000.
13. V. Bilet, O. Dovgoshey and M. Küçükaslan, "Uniform boundedness of pretangent spaces, local constancy of metric valued derivatives and strong right upper porosity at a point," *J. Analysis,* vol. 21, pp. 31–55, 2013.
14. O. Dovgoshey and O. Martio, "Tangent spaces to general metric spaces," *Rev. Roumaine Math. Pures. Appl.,* vol. 56, no. 2, pp. 137–155, 2011.
15. A. Lytchak, "Differentiation in metric spaces," *Algebra and Analysis,* vol. 16, no. 6, pp. 128–161, 2004.

