Advances in Analysis, Vol. 3, No. 2, April 2018
https://dx.doi.org/10.22606/aan.2018.32002

Coefficient Inequalities of a Subclass of Starlike Functions Involving q - Differential Operator

K. R. Karthikeyan[1], K. Amarender Reddy[1*] and M. Thirucheran[2]

[1]Department of Mathematics and Statistics, Caledonian College of Engineering, Muscat, Sultanate of Oman.
[2]Department of Mathematics , L. N. College(Autonomous), Ponneri, Tamilnadu, India.
Email: amarenderkommula@gmail.com

Abstract We introduce a new subclass of spiralike biunivalent functions involving q-differential operator. We obtain the coefficient estimates and Fekete-Szegö inequalities for the functions belonging to this class. Relevant connections with various other known classes have been established.

Keywords: Starlike functions, spiralike functions, bi-univalent functions, coefficient inequalities, Fekete-Szegö, symmetric functions.

1 Introduction, Definitions and Preliminaries

Recently, the area of $q-$analysis has attracted serious attention of researchers. The great interest is due to its applications in various branches of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, $q-$difference and $q-$integral equations and in $q-$transform analysis. The generalized $q-$Taylor formula in the fractional $q-$calculus was introduced by Purohit and Raina [19]. The application of $q-$calculus was initiated by Jackson [9,10]. He was the first to develop the $q-$integral and $q-$derivative in a systematic way. Later, geometrical interpretation of the $q-$analysis has been recognized through studies on quantum groups. Simply, the quantum calculus is ordinary classical calculus without the notion of limits. It defines $q-$calculus and $h-$calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Mohammed and Darus [16] studied approximation and geometric properties of these $q-$operators in some subclasses of analytic functions in compact disk. Recently, Purohit and Raina [19,20] have used the fractional $q-$calculus operators in investigating certain classes of functions which are analytic in the open disk, and Purohit [18] also studied these $q-$operators, defined by using the convolution of normalized analytic functions and $q-$hypergeometric functions. A comprehensive study on applications of $q-$calculus in the operator theory may be found in [2]. Ramachandran et al. [21] have used the fractional $q-$calculus operators in investigating certain bound for $q-$starlike and $q-$convex functions with respect to symmetric points.

In univalent function theory, all geometrically defined subclasses do have beautiful analytic characterization defined in terms of differential inequality. So extending the existing subclasses in q-calculus has numerous applications. To provide a unified approach to the study of various properties of certain subclasses of $\mathcal{A}$, we introduce a new class of analytic functions of complex order involving q-derivative of f.

The q-difference operator denoted as $D_q f(z)$ is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \; (f \in \mathcal{A}, \, z \in \mathcal{U} - \{0\}),$$

and $D_q f(0) = f^{'}(0),$ where $q \in (0,1)$. It can be easily seen that $D_q f(z) \to f^{'}(z) \,\text{as}\, q \to 1^{-}$.

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ normalized by the condition $f(0) = f^{'}(0) - 1 = 0$ which is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}). \tag{1}$$

If $f(z)$ is of the form (1), a simple computation yields

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q}\, a_n z^{n-1}, \; (z \in \mathcal{U}). \tag{2}$$

The inverse function of (2) is given by

$$D_q g(w) = 1 - (1+q)a_2 w - (1+q+q^2)a_3 w^2 + 2(1+q)^2 a_2^2 w^2 + \cdots .$$

Let f and g be analytic in the open unit disk $\mathcal{U}$. The function f is subordinate to g written as $f \prec g$ in $\mathcal{U}$, if there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $\mid w(z) \mid < 1$; $(z \in \mathcal{U})$ such that $f(z) = g(w(z))$, $(z \in \mathcal{U})$.

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. The well known example in this class is the Koebe function, $k(z)$ defined by

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n.$$

Also, let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (z \in \mathcal{U}),$$

which are analytic and convex in $\mathcal{U}$ and satisfy the condition

$$Re(p(z)) > 0; \quad (z \in U).$$

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the well known subclasses of $\mathcal{S}$ which are respectively defined as follows.

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha;\, 0 \leq \alpha < 1 \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \, Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha;\, 0 \leq \alpha < 1 \right\}.$$

Using Alexander transform, it follows that $f(z) \in \mathcal{C}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.

One of the very interesting generalization of the function class $\mathcal{S}^*$ is the so called starlike functions of complex order b which satisfies the condition

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z), \quad (f \in \mathcal{A}),$$

where $\phi \in \mathcal{P}$, the class of functions with positive real part and we denote it by $\mathcal{S}_b(\phi)$. Similarly, let $\mathcal{C}_b(\phi)$ denote the class of functions in $\mathcal{A}$ satisfying the condition

$$1 + \frac{1}{b}\frac{zf''(z)}{f(z)} \prec \phi(z), \quad (f \in \mathcal{A}).$$

Note that $\mathcal{S}_b(1 + z/1 - z) = \mathcal{S}_b$ and $\mathcal{C}_b(1 + z/1 - z) = \mathcal{C}_b$ are the classes considered by Nasr and Aouf in [17] and by Wiatrowski in [26]. Our favorite references of the field are [6,7,8] which covers most of the topics in a lucid and economical style.

It is well known that every function $f \in \mathcal{S}$ has an interval f^{-1}, defined by

$$f^{-1}\{f(z)\} = z;\ (z \in \mathcal{U})$$

and

$$f\left\{f^{-1}(w)\right\} = w\,; \qquad \left(\mid w \mid < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{3}$$

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

The Bieberbach conjecture about the coefficient of the univalent functions in the unit disk was formulated by Bieberbach [3] in the year 1916. The conjecture states that for every function $f \in \mathcal{S}$, given by (1), we have $\mid a_n \mid\leq n$ for every n. Strict inequality holds for all n unless f is the Koebe function or one of its rotation. For many years, this conjecture remained as a challenge to mathematicians. After the proof of $\mid a_3 \mid\leq 3$ by Löwner in 1923, Fekete-Szegö surprised the mathematicians with the complicated inequality

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 4\mu - 3 & \text{when } \mu \geq 1 \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) & \text{when } 0 \leq \mu \leq 1 \\ 3\mu - 1 & \text{when } \mu \leq 0, \end{cases}$$

which holds good for all values $0 \leq \mu \leq 1$. Note that this inequality region was thoroughly investigated by Schaefer and Spencer [24].

Keogh and Merkes [12] obtained the following inequalities for the class of convex and starlike functions

$$\left|a_3 - \mu a_2^2\right| \leq \max\left\{\frac{1}{3}, |\mu - 1|\right\}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \max\left\{1, |3 - 4\mu|\right\}$$

respectively. For a class functions in $\mathcal{A}$ and a real (or more generally complex) number μ, the Fekete-Szegö problem is all about finding the best possible constant $C(\mu)$ so that $\left|a_3 - \mu a_2^2\right| \leq C(\mu)$ for every function in $\mathcal{A}$. Many papers have been devoted to this problem see [4,5,13,14,15].

Motivated by the concept introduced by Sakaguchi in [23], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors. In this paper, we introduce a new subclass of spiralike biunivalent functions using subordination and we obtained the estimates of the $\mid a_2 \mid$ and $\mid a_3 \mid$ for the functions belonging to this new subclass.

Definition 1.1. Let $h(z)$ be a convex univalent function with $h(0) = 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_b^\lambda(\beta, s, t, h)$ if and only if it satisfies the analytic condition,

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda}D_qf(z)}{[f(sz) - f(tz)]^{1-\lambda}} - 1\right\}\right] \prec h(z)\cos\beta + i\sin\beta$$

and

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)w]^{1-\lambda}D_qg(w)}{[g(sw) - g(tw)]^{1-\lambda}} - 1\right\}\right] \prec h(w)\cos\beta + i\sin\beta,$$

where ($z \in \mathcal{U}$; $\lambda \geq 0$; $\frac{-\pi}{2} < \beta < \frac{\pi}{2}$; $b \in \mathbb{C} - \{0\}$). and $s, t \in \mathbb{C}$ with $s \neq t, \mid t \mid\leq 1$.

Remark 1.1. On specializing the parameters and the function $h(z)$, we obtain several new and well known subclasses of analytic functions. Here we list a few of them.

1. If we let $\beta = 0$ and $h(z) = 1 + \frac{\gamma-\alpha}{\pi} i\log\left(\frac{1-e^{2\pi i(1-\alpha)\backslash(\gamma-\alpha)z}}{1-z}\right)$. then the class $\mathcal{S}_b^\lambda(\beta, s, t, h)$ reduces to the form
$$\alpha < Re\left\{1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda}D_qf(z)}{[f(sz) - f(tz)]^{1-\lambda}} - 1\right\}\right\} < \gamma.$$
which is analogues to the class introduced by Kuroki and Owa in [11].
2. If we let $q \to 1^-$ and $b = 1$ in $\mathcal{S}_b^\lambda(\beta, s, t, h)$. then the class reduces to the class introduced and studied by Altınkaya and Yalçın in [1].

3. If we set $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in the class $\mathcal{S}_b^\lambda(\beta, s, t, h)$, we have $\mathcal{S}_b^\lambda(\beta, s, t, \alpha)$ and defined as

$$Re\left\{e^{i\beta}\left\{\frac{[(s-t)z]^{1-\lambda} f'(z)}{[f(sz)-f(tz)]^{1-\lambda}}\right\}\right\} > \alpha\cos\beta, z \in \mathcal{U}$$

and

$$Re\left\{e^{i\beta}\left\{\frac{[(s-t)w]^{1-\lambda} g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}\right\}\right\} > \alpha\cos\beta,$$

where $g(w) = f^{-1}(w)$, $s, t \in \mathbb{C}$ with $s \neq t$, $|\, t \,| \leq 1$. $\beta \in (\frac{-\pi}{2}, \frac{-\pi}{2})$ and $\lambda \geq 0$.

Lemma 1.1. *Let the function $\phi(z)$ given by $\phi(z) = \sum_{n=1}^{\infty} \mathcal{B}_n z^n$ be convex in $\mathcal{U}$. If $h(z) \prec \phi(z)$ $(z \in U)$, then $|\, h_n \,| \leq |\, \mathcal{B}_1 \,|$, $n \in \mathcal{N} = \{1, 2, 3, \ldots\}$.*

Lemma 1.2. *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$ and μ is a complex number ,then*

$$|\, c_2 - \mu c_1^2 \,| \leq 2 \max\{1; |\, 2\mu - 1 \,|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

2 Main Results

In this section, we obtain very interesting Fekete-Szegö inequalities for a certain subclass of analytic functions.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 \neq 0$. If $f \in \mathcal{A}$ satisfies the differential inequality*

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} D_q f(z)}{[f(sz)-f(tz)]^{1-\lambda}} - 1\right\}\right] \prec \phi(z)\cos\beta + i\sin\beta. \tag{4}$$

Then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\, b \,|\cos\beta \,|\, B_1 \,|}{|\kappa_1|} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\kappa_1\, \kappa_2 B_1 b\cos\beta\, e^{-i\beta}}{[(1+q)+(\lambda-1)(s+t)]^2}\right|\right\}, \tag{5}$$

where $\kappa_1 = (1+q+q^2) + (\lambda-1)(s^2+st+t^2)$ and

$$\kappa_2 = \left\{\frac{[(1+q)+(\lambda-1)(s+t)](1-\lambda)(s+t) - \frac{\lambda(1-\lambda)}{2}(s+t)^2}{(1+q+q^2)+(\lambda-1)(s^2+st+t^2)} - \mu\right\}.$$

The result is sharp.

Proof. Let $f \in \mathcal{A}$ satisfy (4), then there exist Schwarz function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|\, w(z) \,| < 1$ in $\mathcal{U}$ such that

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} D_q f(z)}{[f(sz)-f(tz)]^{1-\lambda}} - 1\right\}\right] = \phi(w(z))\cos\beta + i\sin\beta. \tag{6}$$

Define $p(z)$ by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{7}$$

Since $w(z)$ is a Schwarz function, it is clear that $Rep(z) > 0$ and $p(0) = 1$. Therefore

$$\begin{aligned}\phi(z) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right)\\ &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \cdots\right]\right)\\ &= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots .\end{aligned} \tag{8}$$

Now by substituting (8) in (6)

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} D_q f(z)}{[f(sz) - f(tz)]^{1-\lambda}} - 1\right\}\right] =$$
$$\left(1 + \frac{B_1 c_1}{2} z + [\frac{B_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{B_2 c_1^2}{4}] z^2 + \cdots\right)\cos\beta + i\sin\beta.$$

From this equation, we obtain

$$e^{i\beta}\frac{1}{b}\left[(\lambda-1)(s+t) + (1+q)\right] a_2 = \frac{B_1 c_1}{2}\cos\beta$$

$$e^{i\beta}\frac{1}{b}\Big\{ \left[(\lambda-1)(s^2+st+t^2) + (1+q+q^2)\right] a_3 - \frac{\lambda(\lambda-1)}{2}(s+t)^2 a_2^2$$
$$+(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]a_2^2\Big\} = \left(\frac{B_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{B_2 c_1^2}{4}\right)\cos\beta.$$

Or, equivalently

$$a_2 = \frac{e^{-i\beta} B_1 c_1 b \cos\beta}{2\left[(1+q) + (\lambda-1)(s+t)\right]},$$

$$a_3 = \frac{e^{-i\beta} b\left(\frac{B_1 C_2}{2} - \frac{B_1 C_1^2}{4} + \frac{B_2 C_1^2}{4}\right)\cos\beta}{(1+q+q^2) + (\lambda-1)(s^2+st+t^2)} -$$
$$\frac{\{[(1+q)+(\lambda-1)(s+t)](1-\lambda)(s+t) - \frac{\lambda(1-\lambda)}{2}(s+t)^2\} a_2^2}{(1+q+q^2)+(\lambda-1)(s^2+st+t^2)}.$$

On simple computation, we have

$$a_3 - \mu a_2^2 = \frac{e^{-i\beta} b(\frac{B_1 C_2}{2} - \frac{B_1 C_1^2}{4} + \frac{B_2 C_1^2}{4})\cos\beta}{(1+q+q^2)+(\lambda-1)(s^2+st+t^2)} -$$
$$\left\{\frac{[(1+q)+(\lambda-1)(s+t)](1-\lambda)(s+t) - \frac{\lambda(1-\lambda)}{2}(s+t)^2}{(1+q+q^2)+(\lambda-1)(s^2+st+t^2)} - \mu\right\} \times$$
$$\left\{\frac{e^{-i\beta} B_1 c_1 b\cos\beta}{2\left[(1+q)+(\lambda-1)(s+t)\right]}\right\}^2.$$

Therefore

$$a_3 - \mu a_2^2 = \frac{B_1 e^{-i\beta} b\cos\beta}{2\kappa_1}\left\{c_2 - \vartheta c_1^2\right\}$$

where

$$\vartheta = \frac{1}{2}\left\{1 - \frac{B_2}{B_1} - \frac{\kappa_1\kappa_2 B_1 b\cos\beta e^{-i\beta}}{\left[(1+q)+(\lambda-1)(s+t)\right]^2}\right\}.$$

On rearranging the terms and taking modulus both sides, our result now follows by application of Lemma1.2. The result is sharp for the functions

$$\left\{\frac{[(s-t)z]^{1-\lambda}D_q f(z)}{[f(sz)-f(tz)]^{1-\lambda}}\right\} = \phi(z^2)$$

and

$$\left\{\frac{[(s-t)z]^{1-\lambda}D_q f(z)}{[f(sz)-f(tz)]^{1-\lambda}}\right\} = \phi(z).$$

This completes the proof of the Theorem 2.1. □

Corollary 2.2. *Let $\phi(z) = 1+B_1z+B_2z^2+\cdots$. with $B_1 \neq 0$. If $f \in \mathcal{A}$ satisfies the differential inequality*

$$\alpha \leq Re\left\{1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\right\} < \gamma. \tag{9}$$

Then

$$| a_3 - \mu a_2^2 | \leq \frac{(\beta-\alpha)}{\sqrt{2}\,\pi}\sqrt{1-\cos\left(\frac{n(1-\alpha)}{\beta-\alpha}\right)}\max\left\{1; \frac{B_2}{B_1}+(1-2\mu)bB_1\right\}.$$

The result is sharp.

Proof. Let

$$\phi(z) = 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,z}{1-z}\right).$$

Clearly, it can be seen that $\phi(z)$ maps $\mathcal{U}$ onto a convex domain conformally and is of the form

$$h(z) = 1+\sum_{n=1}^{\infty} B_n z^n$$

where $B_n = \frac{\beta-\alpha}{n\pi}\, i\left(1-e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right)$. From the equivalent subordination condition proved by Kuroki and Owa in [11], the inequality (9) can be rewritten in the form

$$1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \phi(z).$$

Following the steps as in Theorem 2.1, we get the desired result. □

Corollary 2.3. *[25] Let $\phi(z) = 1+B_1z+B_2z^2+\cdots$. with $B_1 \neq 0$. If f satisfies the following subordination condition*

$$1+\frac{1}{b}\left[\frac{zD_q f(z)}{f(z)}-1\right] \prec \phi(z)\ (b \in \mathcal{C}-\{0\}),$$

then

$$| a_3 - \mu a_2^2 | \leq \frac{| B_1 b |}{([3]_q-1)}\max\left\{1; \left|\frac{B_2}{B_1}+\frac{B_1 b}{[2]_q-1}\left(1-\frac{[3]_q-1}{[2]_q-1}\mu\right)\right|\right\}.$$

The result is sharp.

Proof. The result follows if we let $\beta = 0$, $\lambda = 0$, $t \to 0$ and $s \to 1$ in Theorem3.1 The result sharp for the function

$$\frac{zD_q f(z)}{f(z)} = \phi(z^2) \quad \text{and} \quad \frac{zD_q f(z)}{f(z)} = \phi(z).$$

□

Taking $q \to 1^-$ in the corollary 2.3, we obtain the Fekete szegö inequality for functions belonging to the class of starlike function of complex order b.

Corollary 2.4. *(See Ravichandran et al. [22]) Let $\phi(z) = 1+B_1z+B_2z^2+\cdots$. with $B_1 \neq 0$. If $f(z)$ belongs to the class of starlike function of complex order b . Then*

$$| a_3 - \mu a_2^2 | \leq \frac{| B_1 || b |}{2}\max\left\{1; \left|\frac{B_2}{B_1}+(1-2\mu)B_1 b\right|\right\}.$$

The result is sharp.

3 Coefficient Inequalities Of Biunivalent Functions

We begin this section with finding the coefficient estimates of $\mathcal{S}_b^{\lambda}(\beta, s, t, h)$.

Theorem 3.1. *Let $f(z)$ be of the form (1) and suppose that $f(z)$ is in the class $\mathcal{S}_b^{\lambda}(\beta, s, t, h)$. then*

$$|\, a_2 \,| \leq \sqrt{\frac{2b \mid B_1 \mid \cos\beta}{\mid 2(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)] + 2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)] - \lambda(\lambda-1)(s+t)^2 \mid}}$$

and

$$|\, a_3 \,| \leq \frac{2b \mid B_1 \mid \cos\beta}{\mid 2(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)] + 2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)] - \lambda(\lambda-1)(s+t)^2 \mid}$$

Proof. Let $f \in \mathcal{S}_b^{\lambda}(\beta, s, t, h)$ and g denote that inverse of f to $\mathcal{U}$. It follows from the Definition1.1 that there exist functions $p(z)$, $q(z) \in \mathcal{P}$(the class of function with positive real part), such that

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)z]^{1-\lambda} D_q f(z)}{[f(sz)-f(tz)]^{1-\lambda}} - 1\right\}\right] = p(z)\cos\beta + i\sin\beta \tag{10}$$

and

$$e^{i\beta}\left[1 + \frac{1}{b}\left\{\frac{[(s-t)w]^{1-\lambda} D_q g(w)}{[g(sw)-g(tw)]^{1-\lambda}} - 1\right\}\right] = q(w)\cos\beta + i\sin\beta \tag{11}$$

$$\left[s, t \in \mathcal{C} \text{ with } s \neq t,\ \mid t \mid \leq 1;\ b \in \mathbb{C} - \{0\};\ \lambda \geq 0,\ \beta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

where $p(z) \prec h(z)$ and $q(w) \prec g(w)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots .$$

respectively. It follows from (10) and (11), we deduce

$$e^{i\beta}\frac{1}{b}[(\lambda-1)(s+t)+(1+q)]a_2 = p_1\cos\beta, \tag{12}$$

$$\begin{aligned} e^{i\beta}\frac{1}{b}\Big\{&[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]a_3 - \frac{\lambda(\lambda-1)}{2}(s+t)^2 a_2^2 \\ &+(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]a_2^2\Big\} = p_2\cos\beta, \end{aligned} \tag{13}$$

and

$$-\,e^{i\beta}\frac{1}{b}[(\lambda-1)(s+t)+(1+q)]a_2 = q_1\cos\beta, \tag{14}$$

$$\begin{aligned} e^{i\beta}\frac{1}{b}\Big\{&2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]a_2^2 - \frac{\lambda(\lambda-1)}{2}(s+t)^2 a_2^2 \\ &+(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]a_2^2 - [(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]a_3\Big\} = q_2\cos\beta. \end{aligned} \tag{15}$$

From (12) and (14) we obtain

$$p_1 = -q_1.$$

By adding (13) and (15), we get

$$e^{i\beta}\frac{1}{b}\left\{2(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]\right.$$
$$\left.-\lambda(\lambda-1)(s+t)^2\right\}a_2^2=(p_2+q_2)\cos\beta. \tag{16}$$

Since $p,\, q \in h(\mathcal{U})$, applying Lemma1.1, we have

$$|\ p_m\ |=|\ \frac{p^m(0)}{m!}\ |\leq|\ \mathcal{B}_1\ |,\ m\in\mathcal{N} \tag{17}$$

and

$$|\ q_m\ |=|\ \frac{q^m(0)}{m!}\ |\leq|\ \mathcal{B}_1\ |,\ m\in\mathcal{N}. \tag{18}$$

Applying (17), (18) and Lemma1.1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|\ a_2\ |\leq\sqrt{\frac{2b\ |\ B_1\ |\cos\beta}{|\ 2(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]-\lambda(\lambda-1)(s+t)^2\ |}}.$$

Subtracting (15) from (13) we have

$$e^{i\beta}\frac{1}{b}\left\{2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]a_3-2[(\lambda-1)(s^2+st+t^2)\right.$$
$$\left.+(1+q+q^2)]a_2^2\right\}=(p_2-q_2)\cos\beta. \tag{19}$$

Or, equivalently

$$a_3=\frac{e^{-i\beta}b(p_2+q_2)\cos\beta}{2(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]-\lambda(\lambda-1)(s+t)^2}$$
$$+\frac{e^{-i\beta}b(p_2-q_2)\cos\beta}{2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]}.$$

Applying (17), (18) and Lemma1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|\ a_3\ |\leq\frac{2b\ |\ B_1\ |\cos\beta}{|\ 2(\lambda-1)(s+t)[(1+q)+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+(1+q+q^2)]-\lambda(\lambda-1)(s+t)^2\ |}.$$

This completes the proof of Theroem3.1. □

Remark 3.1. We note that all the results of Altınkaya and Yalçın[1] can be obtained if we let $q\to 1^-$ and $b=1$ in Theorem3.1.

4 Conclusion

We have obtained the upper bound for the initial coefficients of a subclass biunivalent functions. Various well-known and new results could be obtained as a special case of our results. Since we have defined a class involving $q-$ derivative of f, interesting q- analogue of the Sălăgean, Hohlov and Dziok-Srivastava operators could be defined and used to study various subclasses of the analytic functions. It would be more interesting to extend our study to the class of non-analytic functions.

Acknowledgments. The authors thank the referee for valuable comments and suggestions on the earlier version of this paper. Particularly, we thank for the comments leading to an improvement in the introduction and preliminaries.

References

1. Ş. Altınkaya and S. Yalçın, Initial coefficient bounds for a comprehensive subclass of Sakaguchi type functions, Acta Univ. M. Belii, ser. Math. **23** (2015), 27–33.
2. A. Aral, V. Gupta and R. P. Agarwal, *Applications of q-calculus in operator theory*, Springer, New York, 2013.
3. L. Bieberbach, Über die koeffinzienten derjenigen potenzreihen, welche cine schlichte Abbildung des Einheitskreises wermitteln. S.-B. Preuss. Akad. Wiss. (1916), 940–955.
4. A. Chonweerayoot, D. K. Thomas and W. Upakarnitikaset, On the Fekete-Szegő theorem for close-to-convex functions, Publ. Inst. Math. (Beograd) (N.S.) **52(66)** (1992), 18–26.
5. M. Darus and D. K. Thomas, The Fekete-Szegő theorem for strongly close-to-convex functions, Sci. Math. **3** (2000), no. 2, 201–212.
6. P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.
7. A. W. Goodman, *Univalent functions. Vol. II*, Mariner Publishing Co., Inc., Tampa, FL, 1983.
8. I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Monographs and Textbooks in Pure and Applied Mathematics, 255, Marcel Dekker, Inc., New York, 2003.
9. F. H. Jackson, On $q-$definite integrals, Quarterly J. Pure Appl. Math., **41** (1910),193–203.
10. F. H. Jackson, On $q-$functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, **46** (1908),253–281.
11. K. Kuroki and S. Owa, Notes on new class for certain analytic functions, RIMS Kokyuroku, **1772** (2011) pp. 21âĂŞ-25.
12. F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. **20** (1969), 8–12.
13. W. Koepf, On the Fekete-Szegő problem for close-to-convex functions, Proc. Amer. Math. Soc. **101** (1987), no. 1, 89–95.
14. W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on complex Analysis, Z. Li, F Ren, L Lang , and S. Zhang (Eds), International press Boaton, Mass, USA, (1994), 157-169.
15. W. Ma and D. Minda, Coefficient inequalities for strongly close-to-convex functions, J. Math. Anal. Appl. **205** (1997), no. 2, 537–553.
16. A. Mohammed and M. Darus, A generalized operator involving the q-hypergeometric function, Mat. Vesnik **65** (2013), no. 4, 454–465.
17. M. A. Nasr and M. K. Aouf, Starlike function of complex order, J. Natur. Sci. Math. **25** (1985), no. 1, 1–12.
18. S. D. Purohit, A new class of multivalently analytic functions associated with fractional q-calculus operators, Fract. Differ. Calc. **2** (2012), no. 2, 129–138.
19. S. D. Purohit and R. K. Raina, Certain subclasses of analytic functions associated with fractional q-calculus operators, Math. Scand. **109** (2011), no. 1, 55–70.
20. S. D. Purohit and R. K. Raina, Fractional q-calculus and certain subclasses of univalent analytic functions, Mathematica **55(78)** (2013), no. 1, 62–74.
21. C. Ramachandran, D. Kavitha and T. Soupramanien, Certain bound for q-starlike and q-convex functions with respect to symmetric points, Int. J. Math. Math. Sci. **2015**, Art. ID 205682, 7 pp.
22. V. Ravichandran et al., Certain subclasses of starlike and convex functions of complex order, Hacet. J. Math. Stat. **34** (2005), 9–15.
23. K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan **11** (1959), 72–75.
24. A. C. Schaeffer and D. C. Spencer, *Coefficient Regions for Schlicht Functions*, American Mathematical Society Colloquium Publications, Vol. 35, Amer. Math. Soc., New York, NY, 1950.
25. T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, J. Math. Inequal. **10** (2016), no. 1, 135–145.
26. P. Wiatrowski, The coefficients of a certain family of holomorphic functions, Zeszyty Nauk. Uniw. Łódz. Nauki Mat. Przyrod. Ser. II No. 39 Mat. (1971), 75–85.

