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Unique Existence of Solutions to Initial Value Problem for Differential Equation of Variable-Order

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Abstract Some experts claim that the Riemann-Liouville variable-order fractional integral didn't have semigroup property. This property brought us extreme difficulties when we consider the unique existence of solutions of variable-order fractional differential equations. In this work, based on some analysis technique, by means of fixed point theorem, we consider the existence of solutions to an initial value problem for differential equations of variable-order involving with variable-order fractional integral.

Keywords: Variable-order fractional derivative, initial value problem, fractional differential equations, solution, fixed point theorem

1 Introduction

In this paper, we consider the following initial value problem for variable-order differential equation involving with variable-order fractional integral

$$\begin{cases} D_{0+}^{p(t,x(t))}x(t) = f(t, I^{q-p(t,x(t))}x), \quad 0 < t < +\infty, \\ I_{0+}^{1-p(t,x(t))}|_{t=0} = x_0 \in R, \end{cases} \tag{1.1}$$

where $0 < p(t, x(t)) < 1$, $q \geq 1$, $0 \leq t < +\infty, x \in R$, $D_{0+}^{p(t,x(t))}$ denotes derivative of variable-order defined by

$$D_{0+}^{p(t,(t))}x(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-p(s,x(s))}}{\Gamma(1-p(s,x(s)))}x(s)ds, t > 0, \tag{1.2}$$

and

$$I_{0+}^{q-p(t,x(s))}x(t) = \int_0^t \frac{(t-s)^{q-p(s,x(s))-1}}{\Gamma(q-p(s,x(s)))}x(s)ds, \quad t > 0 \tag{1.3}$$

denotes integral of variable-order $q - p(t)$.

The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in different research areas and engineering, such as physics, chemistry, control of dynamical systems etc. The variable-order fractional derivative is an extension of constant order fractional derivative. In recent years, the operator and differential equations of variable-order have been applied in engineering more and more frequently, for the examples and details, see [1-9].

Although the existing literature on solutions of fractional differential equations is quite wide, few papers deal with the existence of solutions to differential equations involving with variable-order derivative. According to (1.1), (1.2) and (1.3), it is obviously that when $q(t)$ is a constant function, i.e. $q(t) \equiv q$ (q is a finite positive constant), then $I_{0+}^{q(t)}, D_{0+}^{q(t)}$ are the usual Riemann-Liouville fractional integral and derivative [10].

The following properties of fractional calculus operators D_{0+}^q, I_{0+}^q play an important part in discussing the existence of solutions of fractional differential equations.

Proposition 1.1. [10] The equality $I_{0+}^{\gamma}I_{0+}^{\delta}f(t) = I_{0+}^{\gamma+\delta}f(t)$, $\gamma > 0, \delta > 0$ holds for $f \in L(0, b), 0 < b < +\infty$.

Proposition 1.2. [10] The equality $D_{0+}^{\gamma}I_{0+}^{\gamma}f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0, b), 0 < b < +\infty$.

Proposition 1.3. [10] Let $0 < \alpha \le 1$. Then the differential equation

$$D_{0+}^{\alpha} u = 0$$

has unique solution

$$u(t) = ct^{\alpha-1}, c \in R.$$

Proposition 1.4. [10] Let $0 < \alpha \le 1$, $u(t) \in L(0,b)$, $D_{0+}^{\alpha} u \in L(0,b)$. Then the following equality holds

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + ct^{\alpha-1}, c \in R.$$

We are interested in whether the above properties of fractional calculus operators remain true for the operators of variable-order.

Let's take Proposition 1.1 for example.

Example 1.1. Let $p(t) = t$, $q(t) = 1$, $f(t) = 1, 0 \le t \le 3$. Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=1}$.

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{s-1}}{\Gamma(s)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds = \int_0^1 \frac{(1-s)^{s-1} s}{\Gamma(s)} ds \approx 0.472.$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^s}{\Gamma(s+1)} ds = \int_0^1 \frac{(1-s)^s}{s\Gamma(s)} ds \approx 0.686.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \ne I_{0+}^{p(t)+q(t)} f(t)|_{t=1}.$$

So, we see that Propositions 1.1-1.4 are not true for the operators of variable-order. But, for integral of variable-order defined by (1.3), we can find that the index law holds for constant order q and variable-order $p(t)$.

Lemma 1.1. Let $x(t), p(t, x(t))$ be real functions, q be positive constant such that the integrals $I_{0+}^{p(t,x(t))} x(t)$ and $I_{0+}^{q+p(t,x(t))} x(t)$ exist. Then the following expression hold

$$I_{0+}^{q} I_{0+}^{p(t,x(t))} x(t) = I_{0+}^{q+p(t,x(t))} x(t), \quad t > 0. \tag{1.4}$$

Proof. By (1.3), we have that

$$\begin{aligned} I_{0+}^{q} I_{0+}^{p(t,x(t))} x(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s \frac{(s-\tau)^{p(\tau,x(\tau))-1}}{\Gamma(p(\tau,x(\tau)))} x(\tau) d\tau ds \\ = & \frac{1}{\Gamma(q)} \int_0^t \int_\tau^t (t-s)^{q-1} \frac{(s-\tau)^{p(\tau,x(\tau))-1}}{\Gamma(p(\tau,x(\tau)))} x(\tau) ds d\tau \\ = & \frac{1}{\Gamma(q)} \int_0^t \int_0^1 \frac{(t-\tau)^{q-1+p(\tau,x(\tau))}}{\Gamma(p(\tau,x(\tau)))} (1-r)^{q-1} r^{p(\tau,x(\tau))-1} x(\tau) dr d\tau \\ = & \frac{1}{\Gamma(q)} \int_0^t \frac{(t-\tau)^{q-1+p(\tau,x(\tau))}}{\Gamma(p(\tau,x(\tau)))} \beta(q, p(\tau,x(\tau))) x(\tau) d\tau \\ = & \frac{1}{\Gamma(q)} \int_0^t \frac{(t-\tau)^{q-1+p(\tau,x(\tau))}}{\Gamma(p(\tau,x(\tau)))} x(\tau) \frac{\Gamma(q)\Gamma(p(\tau,x(\tau)))}{\Gamma(q+p(\tau,x(\tau)))} d\tau \\ = & \int_0^t \frac{(t-\tau)^{q-1+p(\tau,x(\tau))}}{\Gamma(q+p(\tau,x(\tau)))} x(\tau) d\tau \end{aligned}$$

$$= I_{0+}^{q+p(t,x(\tau))}x(t),$$

which implies that (1.4) holds.

Remark 1.1. But, according to Example 1.1, for $x(t), p(t, x(t))$ be real functions, q is positive constant such that the integrals $I_{0+}^{p(t,x(t))}x(t)$ and $I_{0+}^{q+p(t,x(t))}x(t)$ exist. But, in general, we know that

$$I_{0+}^{p(t,x(t))}I_{0+}^{q}x(t) \neq I_{0+}^{q+p(t,x(t))}x(t), \tag{1.5}$$

for some point t.

Based on characters of calculus of variable-order, here, we don't transform problem (1.1) to an integral equation, but, by means of some analysis techniques, we are able to consider existence and uniqueness of positive solution to (1.1).

Hence, one can not transform a differential equation of variable-order into an equivalent integral equation without these propositions. It is a difficulty for us in dealing with the problems of differential equations, which only involve variable-order derivative. But, we can consider such problems that contains the variable-order integral in nonlinear term.

Definition 1.1. A function $x(t)$ is an unique solution of problem (1.1), if its variable-order fractional integral $I_{0+}^{1-p(t,x(t))}x(t)$ exists uniquely on $[0, +\infty)$, and satisfies problem (1.1).

2 Existence and Unique Result

In this section, we present our main results. The following results will play a very important role in our next discussion.

Lemma 2.1. Let

$$E = \left\{ x(t) \middle| \, x(t) \in C[0, +\infty), \sup_{t\geq 0} \frac{x(t)}{1+t^2} < \infty \right\}$$

with the norm

$$\|x\|_E = \sup_{t\geq 0} \frac{|x(t)|}{1+t^2}.$$

Then, $(E, \|\cdot\|_E)$ is a Banach space.

Proof. The proof is similar to Lemma 2.2 [11]. Here, we omit it. □

We assume that:

(H_1) Let $p : [0, +\infty) \times R \to (0, 1)$ a continuous function;

(H_2) For function $h \in C[0, +\infty)$, there exists function $x : [0, +\infty) \to R$ satisfying equation $I_{0+}^{1-p(t,x(t))}x(t) = h(t), 0 \leq t < \infty$;

(H_3) $t^r f : [0, \infty) \times R \to R$ $(\max\{0, q-2\} \leq r < 1)$ is continuous function, and that there exists constant $L > 0$ satisfying

$$\frac{L}{\Gamma(q)(q-r)} < 1,$$

such that

$$t^r|f(t, (1+t^2)x) - f(t, (1+t^2)y)| \leq \frac{L}{1+t^2}|x-y|, \quad 0 \leq t < +\infty, \ x, y \in R.$$

And let $f(t, 0)$ satisfying

$$\lim_{t\to+\infty} \frac{1}{1+t^2}\int_0^t (t-s)^{\rho-1}|f(s, 0)|ds < +\infty.$$

Theorem 2.1. Assume the conditions (H_1)-(H_3) hold, then problem (1.1) has one unique solution.

Proof. According to the definition of variable-order derivative (1.2) and the Riemann-Liouville fractional derivative, applying integral I_{0+}^{1} on both sides of equation of (1.1), we get

$$I_{0+}^{1-p(t,x(t))}x(t) = x_0 + I_{0+}^{1}f(t, I_{0+}^{q-p(t,x(t))}x). \tag{2.1}$$

It follows from Lemma 1.1 and (2.1) that

$$I_{0+}^{1-p(t,x(t))}x(t) = x_0 + I_{0+}^{1}f(t, I_{0+}^{q-p(t,x(t))}x) = x_0 + I_{0+}^{1}f(t, I_{0+}^{q-1}I_{0+}^{1-p(t,x(t))}x). \tag{2.2}$$

We let $I_{0+}^{1-p(t,x(t))}x(t) = y(t)$ and

$$x_0 = y(0) \doteq y_0, \tag{2.3}$$

then, by (2.2), we could get the following integral equation

$$y(t) = y_0 + \int_0^t f(s, I_{0+}^{q-1}y)ds, 0 \le t < +\infty. \tag{2.4}$$

According the above analysis, we will consider the unique existence result of solution to integral equation (2.4).

Define the operator $T : E \to E$ by

$$Ty(t) = y_0 + \int_0^t f(s, I_{0+}^{q-1}y)ds. \tag{2.5}$$

We assert that the operator $T : E \to E$ is well defined. First of all, we verify that $Ty(t) \in C[0, +\infty)$ for $y \in E$. In fact, let

$$g(t, I_{0+}^{q-1}y) = t^r f(t, I_{0+}^{q-1}y),$$

by (H_3), we know that $g : [0, +\infty) \times R \to R$ is continuous.

For the case of $t_0 \in [0, +\infty)$, take $t > t_0$, $t - t_0 < 1$, then

$$\begin{aligned} & |\int_0^t f(s, I_{0+}^{q-1}y)ds - \int_0^{t_0} f(s, I_{0+}^{q-1}y)ds| \\ = \ & |t^{1-r}\int_0^1 \tau^{-r}g(t\tau, I_{0+}^{q-1}y(t\tau)d\tau - t_0^{1-r}\int_0^1 \tau^{-r}g(t_0\tau, I_{0+}^{q-1}y(t_0\tau)d\tau| \\ \le \ & (t^{1-r} - t_0^{1-r})\int_0^1 \tau^{-r}g(t\tau, I_{0+}^{q-1}y(t\tau)d\tau + t_0^{1-r}\int_0^1 \tau^{-r}(g(t\tau, I_{0+}^{q-1}y(t\tau) - g(t_0\tau, I_{0+}^{q-1}y(t_0\tau))d\tau, \end{aligned}$$

by the continuity of $g(t\tau, I_{0+}^{q-1}y(t\tau))$ and t^{1-r}, we obtain

$$\int_0^t f(s, I_{0+}^{q-1}y)ds \qquad \text{is continuous on point} \ \ t_0.$$

In view of the arbitrariness of t_0, we have

$$y_0 + \int_0^t f(s, I_{0+}^{q-1}y)ds \in C[0, +\infty).$$

On the other hand, for $y \in E$, we have

$$\begin{aligned} |\frac{Ty(t)}{1+t^2}| \le \ & \frac{|y_0|}{1+t^2} + |\frac{1}{1+t^2}\int_0^t f(s, I_{0+}^{q-1}y(s))ds| \\ \le \ & \frac{|y_0|}{1+t^2} + \frac{L}{\Gamma(q-1)(1+t^2)}\int_0^t s^{-r}(1+s^2)^{-1}\int_0^s (s-\tau)^{q-2}|y(\tau)|d\tau ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1+t^2}\int_0^t s^{-r}|f(s,0)|ds \\
\le\ & \frac{|y_0|}{1+t^2}+\frac{L}{\Gamma(q-1)(1+t^2)}\int_0^t s^{-r}\int_0^s (s-\tau)^{q-2}|(1+\tau^2)^{-1}y(\tau)|d\tau ds \\
&+\frac{1}{1+t^2}\int_0^t s^{-r}|f(s,0)|ds \\
\le\ & \frac{|y_0|}{1+t^2}+\frac{L}{\Gamma(q-1)(1+t^2)}\int_0^t s^{-r}\int_0^s (s-\tau)^{q-2}\|y\|_E d\tau ds \\
&+\frac{1}{1+t^2}\int_0^t s^{-r}|f(s,0)|ds \\
=\ & \frac{|y_0|}{1+t^2}+\frac{L\|y\|_E}{\Gamma(q)(1+t^2)}\int_0^t s^{q-r-1}ds+\frac{1}{1+t^2}\int_0^t s^{-r}|f(s,0)|ds \\
=\ & \frac{|y_0|}{1+t^2}+\frac{L\|y\|_E}{\Gamma(q)(q-r)}\frac{t^{q-r}}{1+t^2}+\frac{1}{1+t^2}\int_0^t s^{-r}|f(s,0)|ds,
\end{aligned}
$$

which implies that

$$\sup_{t\ge 0}\frac{Ty(t)}{1+t^2}<\infty.$$

Hence, we obtain that operator $T: E\to E$ is well defined.

Now, for $x, y\in E$, we have

$$
\begin{aligned}
\left|\frac{Tx(t)-Ty(t)}{1+t^2}\right| \le\ & |\frac{1}{1+t^2}\int_0^t (f(s, I_{0+}^{q-1}x(s))-f(s, I_{0+}^{q-1}y(s)))ds| \\
\le\ & \frac{1}{1+t^2}\int_0^t |f(s, I_{0+}^{q-1}x(s))-f(s, I_{0+}^{q-1}y(s))|ds \\
\le\ & \frac{L}{1+t^2}\int_0^t s^{-r}(1+s^2)^{-1}|I_{0+}^{q-1}x(s)-I_{0+}^{q-1}y(s)|ds \\
\le\ & \frac{L}{\Gamma(q-1)(1+t^2)}\int_0^t s^{-r}(1+s^2)^{-1}\int_0^s (s-\tau)^{q-2}|x(\tau)-y(\tau)|d\tau ds \\
\le\ & \frac{L}{\Gamma(q-1)(1+t^2)}\int_0^t s^{-r}\int_0^s (s-\tau)^{q-2}(1+\tau^2)^{-1}|x(\tau)-y(\tau)|d\tau ds \\
\le\ & \frac{Lt^{q-r}}{\Gamma(q)(q-r)(1+t^2)}\|x-y\|_E \\
\le\ & \frac{L}{\Gamma(q)(q-r)}\|x-y\|_E
\end{aligned}
$$

According to $\frac{L}{\Gamma(q)(q-r)}<1$, the Banach contraction principle assures that operator T has one unique fixed point $y^*\in E$, which is unique solution of integral equation (2.4). That is , $y^*\in E$ satisfies

$$y^*(t)=y_0+\int_0^t f(s, I_{0+}^{q-1}y^*)ds, 0\le t<+\infty.$$

Now, by (H_2), for $y^* \in E$, there exits function $x : [0, +\infty)$ satisfying

$$I_{0+}^{1-p(t,x(t))}x(t) = y^*(t), 0 \le t < +\infty.$$

then, by (2.4), we know that $I_{0+}^{1-p(t,x(t))}x(t)|_{t=0} = y^*(0) = y_0 = x_0$. Hence, putting $y^*(t) = I_{0+}^{1-p(t,x(t))}x(t)$ into (2.1), we get that

$$I_{0+}^{1-p(t,x(t))}x(t) = x_0 + \int_0^t f(s, I_{0+}^{q-1}I_{0+}^{1-p(t,x(t))}x), 0 \le t < +\infty$$

according to Lemma 1.1, we have that

$$I_{0+}^{1-p(t,x(t))}x(t) = x_0 + \int_0^t f(s, I_{0+}^{q-p(t,x(t))}x)ds, 0 \le t < +\infty$$

which implies that

$$D_{0+}^{p(t,x(t))}x(t) = f(t, I^{q-p(t,x(t))}x), \quad 0 < t < +\infty,$$

thus, we obtain that $x(t)$ is a solution of problem (1.1). □

Remark 2.1. Unfortunately, we couldn't obtain result of Propositions 1.1, 1.2 for variable-order operators $I_{0+}^{1-p(t,x(t))}$ and $D_{0+}^{1-p(t,x(t))}$, therefore, we don't have any way to know which functions space x should belong to, we only know $I_{0+}^{1-p(t,x(t))}x(t) \doteq y^*(t) \in E$.

Remark 2.2. For Theorem 2.1, the condition (H_2) is stronger, but, it is very important in obtaining the unique existence result of solution to problem (1.1). In (H_2), the existence of function $x(t)$ is another important and complex problem, we will investigate it in our following works.

Remark 2.3. According to Theorem 2.1, $y^*(t) = I_{0+}^{1-p(t,x(t))}x(t)$ is unique solution of problem (1.1). And that, by Remark 2.2, we don't know that what is $x(t)$. For the condition (H_2), when $p(t, x(t))$ is a function of variable t, i.e. $p(t, x(t)) = p(t), 0 \le t < +\infty$, we may consider the approximate function of $x(t)$, in this sense, we could call this approximate function as approximate solution of problem (1.1). For example, we let $p : [0, +\infty) \to (0, 1)$ be a continuous function, for arbitrary small positive ε, there exists positive constants $0 < p_i < 1, i = 1, 2, \cdots, 11$, such that

$$|p(t) - p_1| < \varepsilon, \qquad 0 \le t \le 10, \tag{2.6}$$

$$|p(t) - p_2| < \varepsilon, \qquad 10 < t \le 20, \tag{2.7}$$

$$|p(t) - p_3| < \varepsilon, \qquad 20 < t \le 30, \tag{2.8}$$

$$\vdots$$

$$|p(t) - p_{10}| < \varepsilon, \qquad 90 < t \le 100, \tag{2.9}$$

$$|p(t) - p_{11}| < \varepsilon, \qquad 100 < t < +\infty, \tag{2.10}$$

Thus, by $I_{0+}^{1-p(t)}x(t) = I_{0+}^{1-p_1}x(t) = y^*(t), 0 \le t \le 10$, we take function $x_1^*(t) = D_{0+}^{1-p_1}y^*(t)$ as approximate function of $x(t)$ in the interval $[0, 10]$, under the sense of (2.6).

For $10 < t \le 20$, we may consider $I_{0+}^{1-p(t)}x(t), 0 \le t \le 20$ as following

$$\begin{aligned} I_{0+}^{1-p(t)}x(t) = & \int_0^t \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))}x(s)ds \\ = & \int_0^{10} \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)}x(s)ds + \int_{10}^t \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)}x(s)ds, \end{aligned}$$

in the first part above, we consider function $x(s)$ as $x_1^*(t) = D_{0+}^{1-p_1} y^*(t)$ given, as a result, we obtain a deterministic function as following

$$\int_0^{10} \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x(s)ds = \int_0^{10} \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x_1^*(s)ds \doteq h_1(t), 10 < t \leq 20.$$

Thus, from $I_{0+}^{1-p(t)} x(t) = y^*(t)$ and the expression above, when $10 < t \leq 20$, we have that

$$\int_{10}^{t} \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x(s)ds = y^*(t) - h_1(t), 10 < t \leq 20.$$

Hence, we may take function $x_2^*(t) = D_{10+}^{1-p_2}[y^*(t) - h_1(t)], 10 \leq t \leq 20$ as approximate function of $x(t)$ in the interval $(10, 20]$, under the sense of (2.7).

For $20 < t \leq 30$, we may consider $I_{0+}^{1-p(t)} x(t), 0 \leq t \leq 30$ as following

$$\begin{aligned} I_{0+}^{1-p(t)} x(t) &= \int_0^t \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} x(s)ds \\ &= \int_0^{10} \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x(s)ds + \int_{10}^{20} \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x(s)ds + \int_{20}^{t} \frac{(t-s)^{-p_3}}{\Gamma(1-p_3)} x(s)ds, \end{aligned}$$

similar to the previous arguments, in the first part above, we consider function $x(s)$ as $x_1^*(t) = D_{0+}^{1-p_1} y^*(t)$ given, and in the second part above, we consider function $x(s)$ as $x_2^*(t) = D_{10+}^{1-p_2}[y^*(t) - h_1(t)]$ given. As a result, we obtain a deterministic functions as following

$$\int_0^{10} \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x(s)ds = \int_0^{10} \frac{(t-s)^{-p_1}}{\Gamma(1-p_1} x_1^*(s)ds \doteq h_1(t), 20 < t \leq 30,$$

$$\int_{10}^{20} \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x(s)ds = \int_{10}^{20} \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x_2^*(s)ds \doteq h_2(t), 20 < t \leq 30,$$

Thus, from $I_{0+}^{1-p(t)} x(t) = y^*(t)$ and the expressions above, when $20 < t \leq 30$, we have that

$$\int_{20}^{t} \frac{(t-s)^{-p_3}}{\Gamma(1-p_3)} x(s)ds = y^*(t) - h_1(t) - h_2(t), 20 < t \leq 30.$$

Hence, we may take function $x_3^*(t) = D_{20+}^{1-p_3}[y^*(t) - h_1(t) - h_2(t)], 20 \leq t \leq 30$ as approximate function of $x(t)$ in the interval $(20, 30]$, under the sense of (2.8).

By similar arguments above, we may obtain function $x_{11}^*(t), 100 < t < +\infty$ as approximate function of $x(t)$ in the interval $(100, +\infty)$, under the sense of (2.10).

Thus, we take function defined by

$$x(t) = \begin{cases} x_1^*(t), & 0 \leq t \leq 10, \\ x_2^*(t), & 10 < t \leq 2, \\ \cdots, \\ x_{11}^*(t), & 100 < t < +\infty \end{cases}$$

as approximate function of $x(t)$, which is in $I_{0+}^{1-p(t,x(t))} x(t) = y^*(t), 0 \leq t < +\infty$.

3 Conclusion

The variable-order fractional derivative is an extension of constant order fractional derivative. The existence of solutions to some class of differential equations of variable-order is an interesting object. But, loss of some fundamental properties, ones have some difficulties in dealing with the existence of solutions to some problems for differential equations of variable-order. In this article, by means of fixed point theorem, we have considered the existence and unique of solution to some class of differential equations of variable-order, in which there has variable-order fractional integral.

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