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The Extended Golden Section and Time Series Analysis

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Abstract The Golden ratio has played an important role in musical composition, architecture, visual art, science, and increasingly in signal processing [1,2,3]. Underlying many of these applications are several extensions of the golden proportions including the Golden p-Section by Stakhov, the generalized Golden section by Bradley, and others [4,5]. In this paper we review and introduce generalizations of the Golden ratio. We show that there exists a fundamental connection between the limit of two consecutive terms of recursive sequences, the generalized (p, q)-Golden ratio and the Golden ratio generated by the characteristic equation. We apply these generalizations to forecasting financial time series to illustrate one of their applications in signal processing.

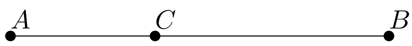
Keywords: Golden Ratio, Euclid, Fibonacci, Aesthetic Ratio, Time Series, Signal Processing

1 Introduction

Traditionally, the Golden ratio has arisen in fields ranging from mathematics to architecture and visual art [6,7,8,9,10,11,12,13,14,15,16,17]. The Golden ratio, also known as the Golden mean or proportion, has appeared in many geometrical constructions, even recently when Janusz Kapusta discovered a new world of geometrical relationships residing within the square and the circle. This marked the first visual construction connecting the Golden and Silver mean proportions in a single diagram [18]. Today the Golden ratio plays an increasing role in engineering and modern physical research [19,20,21,22,23,24,25,26,27,28,29]. Some of its applications in signal processing include:

- Face detection evaluation [1]
- Fashion and textile design [2,3,30]
- Analog-to-digital converter design [31,32]
- Traffic signal timing optimization [33]
- Heart and perception based biometrics [34,35,36]
- Audio and speech sampling [37,38]
- Barcode generation [39]

The Golden ratio was first referenced by Euclid (300 B.C.) in his book "The Elements" [4,40,41]. In this book, Euclid outlined the geometrical problem named the "Division in Extreme and Mean Ratio" [42,43].



Euclid's Theorem. *Divide a line AB into two segments, a larger one CB and a smaller one AC such that:*

$$CB^2 = AB * AC, \tag{1}$$

where $CB > AC$ and $AB = AC + CB$. Dividing both parts of the expression (1) by CB and then by AC, we can rewrite expression (1) in the following form:

$$\frac{CB}{AC} = \frac{AB}{CB}. \tag{2}$$

Equation (2) implies the following algebraic equation:

$$x^2 - x - 1 = 0, \tag{3}$$

where $x = \frac{CB}{AC}$. The positive root of (3) is $\phi \approx 1.618$ and is called the Golden ratio or proportion.

Kepler later discovered that the Golden ratio can be expressed as the ratio of two consecutive Fibonacci numbers [20]. Fibonacci numbers have the property that each term is the sum of the two preceding terms: $f_k = f_{k-1} + f_{k-2}$, $k \geq 2$, where $f_0 = 0$ and $f_1 = 1$.

Extensions of the Golden ratio have been considered by many authors. They can be broadly categorized as:

1. Generalizations of Euclid's theorem [4,5,14,44,45,46].
2. Generalizations of the characteristic equation [47,48,49,4,5,42,50,51].
3. Generalizations of the limit of successive terms recursive sequences [4,52].

In this paper we review and introduce generalizations of the Golden ratio and investigate their properties and application. We examine the generalizations to Euclid's theorem to formulate and present a solution of the extended Euclid problem. We categorize existing recursive sequences and new generalizations of the Golden section using their characteristic equations. We show that the new extension of the Golden sections is generated by the limits of the quotient of two consecutive terms of a recursive sequence. Lastly, we apply the generalizations to analyze financial stock prices.

The benefits and challenges of this work lie in identifying the relationship between Golden ratio, generalizations of Euclid's theorem, the characteristic equation, and the limit of successive terms recursive sequences. The rest of the article is organized as follows: Section 2 introduces a new generalization of the Golden ratio by extending Euclid's problem. Section 3 views this generation based on the characteristic equation generated by homogeneous recurrence relations. Section 4 defines the generalized Golden ratio as a limit of successive terms recursive sequences and investigates its properties. Section 5 summarizes our results and discusses its applications in financial time series analysis.

2 Extending Euclid's Problem: a New Generalization of the Golden Section

In this section we present a review of the generalizations to Euclid's theorem and introduce a new generalization of the extended Euclid problem.

Golden p-Section [4]: Let p be a non-negative integer. Divide line AB into 2 pieces such that:

$$\frac{CB}{AC} = (\frac{AB}{CB})^p, \tag{4}$$

where $CB > AC$ and $AB = AC + CB$. From (4) we obtain:

$$x^{p+1} - x^p - 1 = 0, \tag{5}$$

where $x = \frac{AB}{CB}$. We call the positive root of (5) the Golden p-section or p-ratio. In Table 1, we illustrate other known sequences with varying parameter p [4,50,53]. The ratio of two consecutive p-Fibonacci or p-Lucas sequences decays to the solution of (5).

Table 1. p-Numbers

p	Fibonacci	Lucas
0	$0, 1, 2, 4, 8, 16, 32, 64\dots$	$1, 1, 2, 4, 8, 16, 32, 64\dots$
1	$0, 1, 1, 2, 3, 5, 8, 13\dots$	$2, 1, 3, 4, 7, 11, 18, 29\dots$
2	$0, 1, 1, 1, 2, 3, 4, 6\dots$	$3, 1, 1, 4, 5, 6, 10, 15\dots$
3	$0, 1, 1, 1, 1, 2, 3, 4\dots$	$4, 1, 1, 1, 5, 6, 7, 8\dots$
4	$0, 1, 1, 1, 1, 1, 2, 3\dots$	$5, 1, 1, 1, 1, 6, 7, 8\dots$

Golden (p, q)-Section: Let m be a real number. We generalize the definition of the Golden section by dividing a segment AB into $s + e$ pieces, such that $AB = sAC + eCB$ and:

1. There are s pieces AC of equal length.
2. There are e pieces CB of equal length.
3. Each of the s pieces is shorter than each of the e pieces, or $CB > AC$.
4. The sum of the s AC pieces and e CB pieces is the whole segment AB, or $AB = sAC + eCB$.
5. The ratio of the length of a single larger piece to the smaller piece multiplied by a constant is equal to the power m of the length of the whole segment to that of the larger piece multiplied by a constant:

$$\beta(\frac{CB}{AC}) = (\alpha\frac{AB}{AC})^m, \tag{6}$$

where m, α, β are real (irrational or rational) numbers.

If m is a rational number, $\frac{p}{q}$, where p and q are integers and $q \neq 0$, then (6) has the following form:

$$(\beta\frac{CB}{AC})^q = (\alpha\frac{AB}{CB})^p. \tag{7}$$

Thus, the ratio between the longer piece CB and the shorter one AC raised to the qth power is equal to the ratio between the whole line segment AB and the longer part CB raised to the pth power.

Proposition. *If m is a rational number, $\frac{p}{q}$, where p and q are integers and $q \neq 0$, then* (7) *implies the two algebraic equations:*

1. $\alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0$ for $q = 1$ and $p = 0, 1, 2, 3 \dots$
2. $\beta^q x^{q+1} - \alpha e x - \alpha s = 0$ for $p = 1$ and $q = 1, 2, 3 \dots$

Proof. Let's assume $q = 1$ and $p = 0, 1, 2, 3 \dots$, then equation (7) can be written as:

$$\beta\frac{CB}{AC} = (\alpha\frac{AB}{CB})^p. \tag{8}$$

Denoting

$$x = \frac{AB}{CB}, \tag{9}$$

and using the relationship $AB = sAC + eCB$, we can write:

$$x = \frac{sAC + eCB}{CB} = e + \frac{1}{\frac{CB}{AC}}s. \tag{10}$$

Using (9) we obtain:

$$\frac{CB}{AC} = \frac{1}{\beta}(\alpha x)^p. \tag{11}$$

Substituting (11) into (10), we have:

$$x = e + \frac{s}{\frac{1}{\beta}(\alpha x)^p} \Rightarrow x = e + \frac{\beta s}{(\alpha x)^p} \Rightarrow \alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0. \tag{12}$$

A similar argument can be made for the case $p = 1$, $q = 1, 2, 3 \dots$. Using (7) and $AB = sAC + eCB$ we obtain:

$$(\beta\frac{CB}{AC})^q = \alpha\frac{AB}{CB} = \alpha\frac{sAC + eCB}{CB} = s\frac{AC}{CB} + e = \alpha(s\frac{1}{\frac{CB}{AC}} + e). \tag{13}$$

Denoting $x = \frac{CB}{AC}$, we have:

$$(\beta\frac{CB}{AC})^q = \alpha(s\frac{1}{\frac{CB}{AC}} + e) \Rightarrow (\beta x)^q = \alpha(s\frac{1}{x} + e) \Rightarrow \beta^q x^{q+1} - \alpha e x - \alpha s = 0. \tag{14}$$

□

We call the positive root of these equations:

$$\alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0, \tag{15}$$

$$\beta^q x^{q+1} - \alpha e x - \alpha s = 0 \tag{16}$$

the generalized Golden (p, q)-section or $\phi_{p,q}$. Note that:

1. Every p,q,s, and e in (7) generates its own variant of the division.
2. The definition of the extended Golden (p, q)-section contains the definition of classical Golden section, Bradley's Golden section [5], and Stakhov's Golden p-section [4]. See Table 2.
3. Examples of the generalized Golden (p, q)-section are given below.

$p = 3, q = 4$ $\phi = 1.701$
$p = 2, q = 3$ $\phi = 1.678$
$p = 1, q = 1$ $\phi = 1.618$
$p = 4, q = 3$ $\phi = 1.555$
$p = 3, q = 2$ $\phi = 1.529$

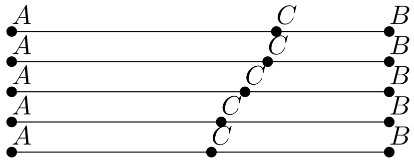


Table 2. Extended sections with $m \in \mathbb{R}$, $\beta = \alpha = 1$

Author	Definition	Equation	p	q	s	e	Reference
Generalized Golden (p, q)-section	$(\beta\frac{CB}{AC})^q = (\alpha\frac{AB}{CB})^p$ $AB = sAC + eCB$	$\alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0$ $\beta^q x^{q+1} - \alpha e x - \alpha s = 0$	$0, 1, 2, 3 \ldots$ 1	1 $1, 2, 3 \ldots$	$\mathbb{R}$	$\mathbb{R}$	New
Euclid's Section	$\frac{CB}{AC} = \frac{AB}{CB}$ $AB = AC + CB$	$x^2 - x - 1 = 0$ $x = \frac{CB}{AC}$	1	1	1	1	[4]
Golden p-Section	$\frac{CB}{AC} = (\frac{AB}{CB})^p$ $AB = AC + CB$	$x^{p+1} - x^p - 1 = 0$	$0, 1, 2, 3 \ldots$	1	1	1	[4]
Bradley's Section	$\frac{CB}{AC} = \frac{AB}{CB}$ $AB = AC + CB$	$x^2 - ex - 1 = 0$	1	1	1	$1, 2, 3 \ldots$	[5]
Multi Parameters Section	$\beta\frac{AB}{AC} = (\alpha\frac{AC}{CB})^p$ $AB = AC + CB$	$x^{p+1} - \alpha x - \beta = 0$	$0, 1, 2, 3 \ldots$	1	1	1	[54]

3 Generalizing the Golden Section Based on the Characteristic Equation

In this section, we review some recursive sequences, most notably Fibonacci and Lucas numbers and their generalizations. We also categorize existing and new generalizations of the Golden section under characteristic equations and recursive relations.

Definition [53]. *A sequence $\{a_n\}$ is said to be defined recursively if its initial values or conditions are specified by $a_0 = C_0, a_1 = C_1 \ldots a_{k-1} = C_{k-1}$ and the sequence terms an are defined by one or more of the previous terms $a_n = \mathbb{R}(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$ for $n > k - 1$ in the sequence:*

$$a_0, a_1, a_2, \ldots, a_{n-1}. \tag{17}$$

Definition [53]. *A linear homogeneous recurrence relation of degree k with constant coefficients $\{c_n\}$ is a relation where each element of a sequence $\{a_n\}$ is a linear combination of previous terms:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}, \tag{18}$$

where $a_0 = C_0, a_1 = C_1 \ldots a_{k-1} = C_{k-1}$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

The degree k can be any number and allows us to make k independent choices (âĂIJdegrees of freedom"). Note a sequence $\{a_n\}$ is unique if it satisfies (18).

Definition. *A relation is a partially homogeneous recurrence relation if some of the constant coefficients c_k are zero (see illustrative examples in Table 4)*

The partial recurrence relation (18) has degree m if a_n can be expressed in terms of some previous $m < k$ terms. In this article, we shall restrict our attention to the second degree homogeneous linear recurrence relations. Table 3 and Table 4 present second degree ($k = 2$) commonly used homogeneous and partially homogeneous recurrences.

Table 3. Second degree ($k = 2$) common homogeneous recurrences

Numbers	Recurrence Relation $k \geq 2$	Initial Conditions	Sequences	Limit of Quotient	Reference
Fibonacci	$f_k = f_{k-1} + f_{k-2}$	$f_0 = 0, f_1 = 1$	$0, 1, 1, 2, 3, 5, 8, 13 \ldots$	$(1 + \sqrt{5})/2$	[14,41,53,55]
Lucas	$l_k = l_{k-1} + l_{k-2}$	$l_0 = 2, l_1 = 1$	$2, 1, 3, 4, 7, 11, 18, 29 \ldots$	$(1 + \sqrt{5})/2$	[14,41,53]
Weighted Fibonacci	$f_k = \alpha f_{k-1} + \beta f_{k-2}$	$f_0 = 0, f_1 = 1$	$(a = 1, b = 1)$ $0, 1, 1, 2, 3, 5, 8, 13 \ldots$	$(\alpha + \sqrt{\alpha^2 + 4\beta})/2$	[50,53]
Weighted Lucas	$l_k = \alpha l_{k-1} + \beta l_{k-2}$	$l_0 = 2, l_1 = 1$	$(a = 1, b = 1)$ $2, 1, 3, 4, 7, 11, 18, 29 \ldots$	$(\alpha + \sqrt{\alpha^2 + 4\beta})/2$	[50,53]
m-Fibonacci	$f_{k-1,m} + f_{k-2,m}$ $m > 0$	$f_0 = 0, f_1 = 1$	$(m = 1)$ $0, 1, 1, 2, 3, 5, 8, 13 \ldots$	$(m + \sqrt{m^2 + 4})/2$	[56]
Pell	$p_k = 2p_{k-1} + p_{k-2}$	$p_0 = 0, p_1 = 1$	$0, 1, 2, 5, 12, 29, 70, 169 \ldots$	$1 + \sqrt{2}$	[56,57]
Pell-Lucas	$p_k = 2p_{k-1} + p_{k-2}$	$p_0 = 2, p_1 = 2$	$2, 2, 6, 14, 34, 82, 198, 478 \ldots$	$1 + \sqrt{2}$	[56]

Table 4. Second degree ($k = 2$) common and new partially homogeneous recurrences

Numbers	Recurrence Relation	Initial Conditions	Sequences	Limit of Quotient	Reference
Padovan	$d_k = d_{k-2} + d_{k-3}$ $k \geq 3$	$d_0 = d_1 = d_2 = 1$	$1, 1, 1, 2, 2, 3, 5, 7 \ldots$	≈ 1.324	[50]
Perrin	$p_k = p_{k-2} + p_{k-3}$ $k \geq 3$	$p_0 = 3, p_1 = 0, p_2 = 2$	$3, 0, 2, 3, 2, 5, 5, 7 \ldots$	≈ 1.324	[50]
p-Fibonacci	$f_k = f_{k-1} + f_{k-p-1}$ $k > p + 1$	$f_0 = 0, f_1 = \ldots = f_{p+1} = 1$	$(p = 2)0, 1, 1, 1, 2, 3, 4, 6 \ldots$	ϕ_p	[4,50,53]
p-Lucas	$l_k = l_{k-1} + l_{k-p-1}$ $k > p + 1$	$l_0 = p + 1, l_1 = \ldots = l_{p+1} = 1$	$(p = 2)3, 1, 1, 1, 4, 5, 6, 10 \ldots$	ϕ_p	[4,50,53]

Solving linear homogeneous recurrences [53]. *To solve the recurrence relation* (18)*, we will guess that the solution has the form $a_n = r^n$. We substitute into the recurrence* (18) *to get:*

$$r^n = c_1 r^{n-1} + c_2 r^{n-k}, \tag{19}$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Dividing both sides of (19) *by r^{n-k} gives:*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0. \tag{20}$$

Equation (20) is known as the characteristic equation of the recurrence relation (18). It is a polynomial of degree k and thus by the Fundamental Theorem of Algebra has k complex roots $r_1, r_2, \ldots, r_k$ counted by multiplicity. The general solution is therefore: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$. More precisely, it can be shown that for $c_1, c_2, \ldots, c_k \in \mathbb{R}$

1. If the characteristic equation (20) has k distinct roots $r_1, r_2, \ldots, r_k$, then the sequence $\{a_n\}$ is a solution of the recurrence relation (18) if and only if:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n, \tag{21}$$

where $n = 0, 1, 2, \ldots$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants that are determined by the k initial conditions $a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$.

2. If the characteristic equation (20) has t roots with multiplicities $m_1, m_2, \ldots, m_t$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation (18) if and only if:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m_1-1})r_1^n + \ldots + (\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{2,m_t-1})r_t^n, \tag{22}$$

where $n = 0, 1, 2, \ldots$ and $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$ that depend on initial conditions. Note the roots of this polynomial are called the characteristic roots of the recurrence relation. The positive root is called the generalized Golden ratio.

Example 1: Consider a second order linear homogeneous recurrence of the form $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0, f_2 = 1$. The characteristic equation of f_n is $x^2 - \alpha x - \beta = 0$. The general form of the solution is $f_n = \alpha\phi^n + \beta\phi'^n$ where α and β are unknowns. Using the initial conditions, $\alpha + \beta = 0$, $\phi\alpha + \phi'\beta = 1$, we get:

$$f^k = \frac{r_1^k - r_2^k}{r_1 - r_2}, \tag{23}$$

where $r_1 = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$ and $r_2 = \frac{\alpha-\sqrt{\alpha^2+4\beta}}{2}$. For the Fibonacci sequence, $\alpha = \beta = 1$ and the characteristic equation is $x^2 - x - 1 = 0$. This gives us the roots $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\phi' = \frac{1-\sqrt{5}}{2} \approx 0.618$. We can also generate Fibonacci and Lucas numbers from Binet's formula:

$$f_n = \frac{\phi^n - (-\frac{1}{\phi})^n}{\sqrt{5}}. \tag{24}$$

Example 2: Consider a second order linear homogeneous recurrence of the form $f_{k,m} = mf_{k-1,m} + f_{k-2,m}$, with $f_0 = 0, f_1 = 1$, and $m > 0$. This m-Fibonacci sequence may be expressed in the form:

$$f_{k,m} = \alpha r_1^k + \beta r_2^k, \tag{25}$$

where $r_1 = \frac{m+\sqrt{m^2+4}}{2}$, $r_2 = \frac{m-\sqrt{m^2+4}}{2}$ α, β are constants. For values $k = 0, 1$ we obtain:

$$f_{0,m} = a + b = 0 \qquad f_{1,m} = ar_1 + br_2, \tag{26}$$

which implies $a = \frac{1}{r_1 - r_2}$ We can use this result to derive Binet's formula:

$$f_{k,m} = ar_1^k + br_2^k = \frac{r_1^k - r_2^k}{r_1 - r_2}. \tag{27}$$

By simplifying (27), we obtain:

$$f_{k,m} = \frac{1}{\sqrt{m^2+4}} \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} m^{k-1-2i} (\sqrt{m^2+4})^{2i+1}. \tag{28}$$

Example 3: Consider a second order linear homogeneous recurrence of the form $g_{k,\phi} = mg_{k-1,\phi} + g_{k-2,\phi}$, where $m = \phi = \frac{1+\sqrt{5}}{2}$. Using a similar procedure to that in previous example, we can derive the general form of this Fibonacci-Golden sequence:

$$g_{k,m} = \frac{1}{\sqrt{\phi^2+4}}(r_1^k - r_2^k), \tag{29}$$

where $r_1 = \frac{\phi+\sqrt{\phi^2+4}}{2}$ and $r_2 = \frac{\phi-\sqrt{\phi^2+4}}{2}$. By simplifying (29), we obtain:

$$g_{k,m} = \frac{1}{\sqrt{\phi^2+4}} \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} \phi^{k-1-2i} (\sqrt{\phi^2+4})^{2i+1}. \tag{30}$$

Example 4: Consider a second order partial linear homogeneous recurrence of the form $f_k = f_{k-1} + f_{k-p-1}$ where $f_0 = f_1 = \ldots = f_{p+1} = 1$ (see Table 1. The characteristic equation of a_n is $x^{p+1} - x^p - 1 = 0$. The root of this equation is called the Golden p-section [4]. For $p = 0, 1, 2, 3$ the roots are .5, .618, .683, .725, respectively. Table 5 summarizes the extensions.

Table 5. Extended Golden sections and their generating equations ($\alpha = \beta = 1$)

Golden section	p	q	e	s	Generating Equation	Root	Reference
Euclid	1	1	1	1	$x^2 - x - 1 = 0$	$\frac{1+\sqrt{5}}{2}$	[53]
Family of Metallic Means	1	1	$\mathbb{Z}^+$	$\mathbb{Z}^+$	$x^2 - sx - e = 0$	$\frac{s+\sqrt{s^2-4e}}{2}$	[47,48,49]
Silver Mean	1	1	2	1	$x^2 - 2x - 1 = 0$	$1 + \sqrt{2}$	[47,48,49]
Bronze Mean	1	1	3	1	$x^2 - 3x - 1 = 0$	$\frac{3+\sqrt{13}}{2}$	[47,48,49]
Metallic Mean	1	1	4	1	$x^2 - 4x - 1 = 0$	$2 + \sqrt{13}$	[47,48,49]
Bradley	1	1	$\mathbb{Z}^+$	1	$x^2 - mx - 1 = 0$	$\frac{m+\sqrt{m^2+4}}{2}$	[5]
Copper Mean	1	1	1	2	$x^2 + x - 2 = 0$	2	[47,48,49]
Nickel Mean	1	1	1	3	$x^2 + x - 3 = 0$	$\frac{1+\sqrt{13}}{2}$	[47,48,49]
Complex	1	1	1	$\frac{3}{2}$	$x^n + x - \frac{3}{2} = 0$	$\frac{1+j\sqrt{5}}{2}$	[58]
Three Roots	2	1	1	1	$x^3 - x^2 - 1 = 0$	≈ 1.466	[4]
Four Roots	3	1	1	1	$x^4 - x^3 - 1 = 0$	≈ 1.380	[4]
Trigonometric Mean	1	1	$2cosx$	1	$x^2 - 2xcosx - 1 = 0$	$1 + \sqrt{cos^2x + 1}$	New
Aesthetic	1	1	-7	11	$x^2 + 7x - 11 = 0$	≈ 1.322	New
Generalized Golden	$\mathbb{Z}^+$	1	$\mathbb{R}$	$\mathbb{R}$	$\alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0$		New
Generalized Golden	1	$\mathbb{Z}^+$	$\mathbb{R}$	$\mathbb{R}$	$\beta^q x^{q+1} - e\alpha x - \alpha s = 0$		New

Properties [40]. *The generalized Golden ratio, $\phi = e + \frac{s}{\phi}$, can be expressed by:*

1. *A series of continued fractions:*

$$\phi = s + \cfrac{e}{s + \cfrac{e}{s + \cfrac{e}{s + \cfrac{e}{s + \cfrac{e}{s + \dots}}}}}. \tag{31}$$

2. *A series of continued square roots:*

$$\phi = \sqrt{s + e\sqrt{s + e\sqrt{s + e\sqrt{s + e\sqrt{s + e \dots}}}}} \tag{32}$$

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4 Generalizing the Golden Section Based on Recursive Sequences

Johannes Kepler [20] observed that the ratio of consecutive Fibonacci numbers converges to the Golden ratio. In this section, we present a review of existing and new generalized Golden sections generated by the limits of the quotient of two consecutive terms of a recursive sequence [49].

Definition. *A sequence is monotonic if it is either increasing or decreasing.*

Proposition 1. *If the sequence $\{a_n\}$ is generated by a linear monotonic homogeneous degree $k = 2$ recurrence: $a_n = ea_{n-1} + sa_{n-1}$, $n \geq 2$, then:*

1. *The ratio of consecutive $\{a_n\}$ numbers converges to the solution of the following equation: $x^2 - ex - s = 0$.*
2. *If $e = s = 1$, then $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{1+\sqrt{5}}{2}$.*
3. *If $e = -7, s = 11$, then $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} \approx 1.322$. We call this the Aesthetic ratio (see (34)).*
4. *The limit of the sequence is independent of the initial conditions.*

Proposition 2. *If the sequence $\{a_n\}$ is generated by a linear monotonic partial homogeneous degree $k = 2$ recurrence, i.e. $a_n = ea_{n-1} + sa_{n-p-1}$, $n \geq 2$, then:*

1. *The ratio of consecutive $\{a_n\}$ numbers converges to the solution of the following equation: $x^{p+1} - ex^p - s = 0$.*

2. *If* $e = s = 1$, *then* $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \tau_p$. *where* τ_p *is a Fibonacci p-number illustrated in Table 1. The limit of the sequence is independent of the initial conditions.In [4] Stakhov showed that* τ_p *is the solution to this equation.*

Proposition 3. *If the sequence* $\{a_n\}$ *is generated by a linear monotonic homogeneous degree* $k = 2$ *recurrence:* $a_n = ea_{n-m} + sa_{n-k}$, $n \geq 2$, *then:*

1. *The ratio of consecutive* $\{a_n\}$ *numbers converges to the solution of the following equation:*

$$x^{m+k} - ex^k - sx^m = 0. \tag{33}$$

We call the sequences that generate (33) *with real or irrational initial conditions the family of* (k, m) *Fibonacci sequences and the roots of* (33) *the family of generalized ratios.*

2. *The limit of the sequence is independent of the initial conditions.*

Proof.

$$\begin{aligned} x &= \lim_{n\to\infty} \frac{a_n}{a_{n-1}} \\ &= \lim_{n\to\infty} \frac{ea_{n-m} + sa_{n-k}}{a_{n-1}} \\ &= e \lim_{n\to\infty} \frac{1}{\frac{a_{n-1}}{a_{n-m}}} + s \lim_{n\to\infty} \frac{1}{\frac{a_{n-1}}{a_{n-k}}} \\ &= e \lim_{n\to\infty} \frac{1}{\frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{n-m+1}}{a_{n-m}}} + s \lim_{n\to\infty} \frac{1}{\frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{n-k+1}}{a_{n-k}}} \\ &= \frac{e}{x^{m-1}} + \frac{s}{x^{k-1}} \Rightarrow x^{m+k} - ex^k - sx^m = 0. \end{aligned}$$

Propositions 1 and 2 can be proved in a similar fashion. □

The parameters m and k can generate both new and commonly used sequences and their corresponding ratios. For example,

– If $m = k = 1$, then $a_n = a_{n-1}(e + s)$ and $x^2 - ex - sx = 0$. For $a_1 = e = s = 1$, $x = \sqrt{2}$ and we have the base-2 sequence: $2^0, 2^1, 2^2, 2^3, \ldots$.
– If $m = k + 1$, then $x^{2k+1} = ex^k + sx^{k+1} \Rightarrow x^{k+1} = e + sx$. For $k = e = s = 1$, we have the Fibonacci and Lucas sequences whose two consecutive terms converge to the Golden Ratio (see Table 2).
– If $m = k + 1$, $e = s = 1$ and $k = 2$, we have Perrin and Padovan sequences (and their variations) whose two consecutive terms converge to ≈ 1.324 (see Table 4).
– If $m = k + 1$, $e = s = 1$ and $k = 3$, we have new sequences derived from our generalized equation (33) whose two consecutive terms converge to $\lim_{a_n} a_{n-1}$.

Aesthetic Ratio. *For a sequence* $\{a_n\}$ *that is generated by a linear monotonic homogeneous degree* $k = 2$ *recurrence, we designate the Aesthetic ratio as:*

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} \approx 1.322, \tag{34}$$

where $a_n = ea_{n-1} + sa_{n-1}$, $n \geq 2, e = -7, s = 11$. *The Aesthetic ratio derives its name from its use in art. It can be seen on a statistical study of 565 famous works of art by Bellini, Caravaggio, Cesanne, Goya, van Gogh, Delacroix, Rembrandt, and Toulouse-Lautrec [59]. It is the average ratio of the two dimensions of these paintings, as summarized in Table 6.*

Definition. *Let* $\{b_n\}$ *be a monotonic homogeneous or partial homogeneous degree* k *recurrence. We call the quantity:*

$$\phi_n = \lim_{n\to\infty} \frac{b_n}{b_{n-1}}, \tag{35}$$

the generalized Golden section for the sequence $\{b_n\}$.

Table 6. Ratio of dimensions of famous paintings

Artist	Number of Paintings	Average Error
Bellini	53	1.46 ± 0.10
Caravaggio	37	1.32 ± 0.15
Cezanne	100	1.26 ± 0.27
Delacroix	42	1.32 ± 0.17
Van Gogh	69	1.32 ± 0.19
Goya	34	1.04 ± 0.04
Rembrandt	39	1.33 ± 0.14
Toulouse-Lautrec	64	1.36 ± 0.12

Tables 3 and 4 summarizes existing and new generalized Golden sections generated by taking the limit of the quotient of two consecutive terms of a recursive sequence.

Properties. *1. Assume that the sequence $\{f_n\}$ is generated by a second order linear homogeneous recurrence and satisfies* (35)*. For the arbitrary integer constants m and k the following holds:*

$$\lim_{n\to\infty} \frac{f_{n+m}}{f_{n+k}} = \gamma^{m-k} \tag{36}$$

where $m > k$.

2. Assume that the sequence $\{f_n\}$ is generated by a second order linear homogeneous recurrence with the solution (23)*. Then:*

$$\lim_{k\to\infty} \frac{f_k}{f_{k-1}} = r_1 \qquad \lim_{k\to\infty} \frac{f_{k-1}}{f_k} = r_2. \tag{37}$$

Proof. Since $r_1 > r_2$ then $\lim_{k\to\infty} \frac{r_2^k}{r_1^k} = 0$ and $f_k = \frac{r_1^k - r_2^k}{r_1 - r_2}$ we have:

$$\lim_{k\to\infty} \frac{f_k}{f_{k-1}} = \lim_{k\to\infty} \frac{r_1^k - r_2^k}{r_1^{k-1} - r_2^{k-2}} = \frac{1 - \lim_{k\to\infty} \frac{r_2^k}{r_1^k}}{\frac{1}{r_1} - \lim_{k\to\infty} \frac{r_2^k}{r_1^k} \frac{1}{r_1}} = r_1.$$

Using a similar approach, we can show $\lim_{k\to\infty} \frac{f_{k-1}}{f_k} = r_2$. □

Definition. *We say two recursive sequences, $f_k = f(t_{k-1}, t_{k-2})$ and $g_k = g(t_{k-1}, t_{k-2})$, where $k \geq 2$, generate a Golden section power sequence $\phi^1, \phi^2, \ldots, \phi^n$ if*

$$\phi^n = \alpha f_n + \beta g_n \tag{38}$$

where $n \in \mathbb{Z}^+$.

Example 5: Let the roots of the characteristic equation be $x^2 - x - 1 = 0$. It can be shown that the classical Fibonacci and Lucas sequences generate a Golden section power sequence.

$$\phi^n = \frac{l_n + f_n\sqrt{5}}{2} \qquad \phi^{-n} = \frac{l_n - f_n\sqrt{5}}{2} \tag{39}$$

Proof. Proof by mathematical induction.

$$\phi = \frac{1+\sqrt{5}}{2} \quad \phi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{3+\sqrt{5}}{2}.$$

$$\phi^3 = (\frac{1+\sqrt{5}}{2})^3 = (\frac{3+\sqrt{5}}{2})(\frac{1+\sqrt{5}}{2}) = \frac{4+2\sqrt{5}}{2}.$$

$$\phi^4 = (\frac{1+\sqrt{5}}{2})^4 = (\frac{4+2\sqrt{5}}{2})(\frac{1+\sqrt{5}}{2}) = \frac{7+3\sqrt{5}}{2} \Rightarrow \phi^n = \frac{l_n + f_n\sqrt{5}}{2}.$$

A similar argument can be used to show $\phi^{-n} = \frac{l_n - f_n\sqrt{5}}{2}$ □

Example 6: The 2-Fibonacci, $f_k = 2f_{k-1} + f_{k-2}, f_0 = 0, f_1 = 1$, and modified Pell sequences, $p_k = 2p_{k-1} + p_{k-2}, p_0 = 1, p_1 = 3$, generate a Silver mean power sequence

$$\gamma^n = 2 * p_n + p_{n-1} + (2f_n + f_{n-1})\sqrt{2} \quad \gamma^{-n} = 2 * p_n + p_{n-1} - (2f_n + f_{n-1})\sqrt{2}, \tag{40}$$

where $\gamma = 1 + \sqrt{2}$

Proof. Proof by mathematical induction.

$$\gamma = 1 + \sqrt{2} \qquad \gamma^2 = (1 + \sqrt{2})^2 = 3 + 2\sqrt{2}.$$

$$\gamma^3 = (1 + \sqrt{2})^3 = (1 + \sqrt{2})(3 + 2\sqrt{2}) = (2 * 3 + 1) + (2 * 2 + 1)\sqrt{2}.$$

$$\gamma^4 = (1 + \sqrt{2})^4 = (1 + \sqrt{2})(7 + 5\sqrt{2}) = (2 * 7 + 3) + (2 * 5 + 2)\sqrt{2}.$$

$$\Rightarrow \gamma^n = 2p_n + p_{n-1} + (2f_n + f_{n-1})\sqrt{2} \tag{41}$$

A similar argument can be made to show $\gamma^{-n} = 2 * p_n + p_{n-1} - (2f_n + f_{n-1})\sqrt{2}$. □

5 Simulation

This section explores the application of the generalized golden ratio for forecasting financial time series. Forecasting methodologies include correlation analysis [60], moving averaging models [61,62,63], logistic regression [64], artificial neural networks [65,66,67], support vector machines [68], and decision tree analysis [69,70]. This section presents simulation results from applying the generalized golden ratio to one commonly used moving average model - the moving average crossover. This model is chosen because of its:

- Human interpretability with intuitive and transparent inputs and outputs
- Support by academic and industry practitioners
- Extensive use as a filter in DSP that is optimal for reducing random noise while retaining a sharp step response
- Simple and efficient low-pass filtering operation that smooths out price fluctuations
- Application to any universe of stocks

Golden Section Moving Average: The golden (p, q) section N-day moving average is calculated by:

$$y_{p,q}[N] = \frac{1}{\phi_{p,q}} \cdot x[1] + \phi_{p,q} \cdot x[N] + \frac{1}{N} \sum_{k=2}^{N-1} x[k], \tag{42}$$

where $\phi_{p,q}$ is the positive root of (15) and $x[N]$ is the data value at day N.

Moving Average Crossover Model: The model generates a signal when a stock price's short-term moving average crosses its long-term moving average. This cross may indicate that the stock is exhibiting upward (downward) momentum, and thus its price that is moving up (down) in the short term is likely to continue moving up (down) [71].The signal from the generalized golden ratio moving average model is:

$$\mathcal{R} = \begin{cases} BUY & \text{if } y_{p,q}[M] > y_{p,q}[N] \\ SELL & \text{if } y_{p,q}[M] < y_{p,q}[N] \\ HOLD & \text{otherwise} \end{cases}, \tag{43}$$

where M and N are the short and long term moving average periods in days, respectively.

Table 7 provides performance results from using the signal described in [43] to trade SPY (SPDR S&P 500), the exchange traded fund that tracks the S&P 500 stock market index. The results are calculated over a ten year time horizon from January 2007 to July 2017, and short and long term moving averages of $M = 10$ and $N = 50$ are selected to be consistent with industry practice. (p, q) parameters are taken from the examples of the Golden section provided earlier in this paper to generate seperate strategies. The performance of each strategy is evaluated against the benchmark of buying and holding the index over the same time period using a risk-adjusted return measure. Two such often used risk measures are the Sharpe and Sortino ratios. The Sharpe ratio is defined as:

$$\text{Sharpe Ratio} = \frac{\overline{R_p} - R_f}{\sqrt{\sigma_p}}, \tag{44}$$

where $\overline{R_p}$ is the average portfolio return, R_f is the risk free rate, σ_p is the portfolio standard deviation.

The Sharpe ratio captures how well strategy returns compensate the investor for the amount of risk taken. The Sortino ratio is a variation of the Sharpe ratio that only factors downside risk rather than total risk of the portfolio [72]. The greater a strategy's Sharpe or Sortino ratio, the better return for the same risk.

The Golden section strategies outperform the benchmark index robustly across (p, q) parameters with an average Sharpe and Sortino ratios of .56 and 0.78, respectively. The highest Sharpe and Sortino ratios are reached with parameters $p = 2, q = 3$, and Figure 1 shows the corresponding strategy's full backtest performance and moving average series. Backtests are done on the Quantopian platform and take into account transaction costs. Results suggest that the Golden section can be used in moving average indicators to detect price momentum trends.

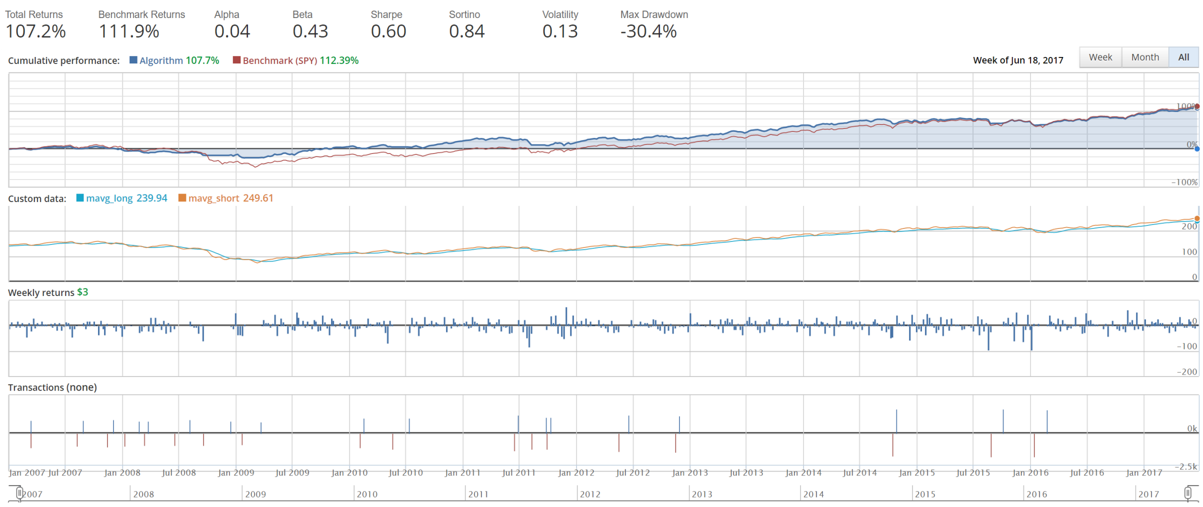

Figure 1. Golden $(2, 3)$-Section MA Signals and Backtest Results

Table 7. Generalized Golden Section Crossover Strategy Performance

(p, q)-Section	ϕ	Sharpe Ratio	Sortino Ratio	Cumulative Return
$(3, 4)$	1.701	0.56	0.78	101%
$(2, 3)$	1.678	0.60	0.84	107%
$(1, 1)$	1.618	0.60	0.83	106%
$(4, 3)$	1.555	0.54	0.73	87%
$(3, 2)$	1.529	0.52	0.71	81%
Benchmark	SPY	0.43	0.60	112%

6 Conclusion

In this paper we review existing variations of the Golden section and introduce generalizations to Euclid's problem. We extend the concepts of Golden ratio from different points of view: through the generalization of Euclid's section definition, the generalization of the characteristic equation, and the definition of limit of successive terms in recursive sequences. Finally, we illustrate the application of generalized Golden sections in detecting stock price trends. Future work will apply higher order degree homogeneous linear recurrence relations to build upon the examined applications in financial time series forecasting.

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