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Common Fixed Point Results in S-metric Spaces

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Abstract S-metric spaces are introduced as a generalization of metric spaces. In this paper, we present some common fixed point theorems about four mappings using the notion of compatible mappings on complete S-metric spaces. We give illustrative examples to verify the obtained result. The results not only directly improve and generalize some fixed point results in S-metric spaces, but also expand and complement some previous results.

Keywords: S-metric space, compatible mappings, common fixed point.

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1 Introduction and Preliminaries

Banach's contraction mapping principle in metric spaces is the most well-known result in the theory of fixed points. When the fixed point theorem was proved by Banach[1] in 1922 for a contraction mapping in a complete metric space, scientists around the world are publishing new results that are connected either to establish a generalization of metric space or to get an improvement of contractive condition. In addition, with the improvement of Banach's contractive conditions, metric spaces are more and more important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in other spaces, such as 2-metric spaces, cone metric spaces, b-metric spaces, G-metric spaces,D-metric spaces and others[2]-[4]. One of the generalizations of metric spaces is given in the paper of Sedghi et al. [5]. They introduced a notion of S-metric spaces and gave some of their properties. For more details on S-metric spaces, one can refer to the paper[7]-[16].

The object of this paper is to get some common fixed point results in the complete S-metric spaces, which were inspired by [14] and [15].

First,The notion of S-metric spaces is defined as follows.

Definition 1.1. *(See [5]) Let X be a nonempty set. A function $S : X^3 \to [0, \infty)$ is said to be an S-metric on X, if for each $x, y, z, a \in X$:*

(S_1) $S(x, y, z) \geq 0$;
(S_2) $S(x, y, z) = 0$ *if and only if* $x = y = z$;
(S_3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 1.2. *(See [5]) Let $X = R^2$ and d be an ordinary metric on X. Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in R^2$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.*

Definition 1.3. *(See [6]) Let X be a nonempty set. A B-metric on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $b \geq 1$ such that the following conditions hold for all $x, y, z \in X$:*

(B_1) $d(x, y) = 0$ *if and only if* $x = y$;
(B_2) $d(x, y) = d(y, x)$;
(B_3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

The pair (X, d) is called a B-metric space.

Proposition 1.4. *(See [7]) Let (X, S) be an S-metric space and let*

$$d(x, y) = S(x, x, y) \tag{1}$$

for all $x, y \in X$. Then we have

(1) *d is a B-metric on X;*
(2) *$x_n \to x$ in (X, S) if and only if $x_n \to x$ in (X, d);*
(3) *$\{x_n\}$ is a Cauchy sequence in (X, S) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).*

Proposition 1.5. *(See [7]) Let (X, S) be an S-metric space. Then we have*

(1) *X is first-countable;*
(2) *X is regular.*

Remark 1.6. *By Propositions 1.4 and 1.5 we have that every S-metric space is topologically equivalent to a B-metric space.*

Lemma 1.7. *(See [5]) Let (X, S) be an S-metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy.*

Definition 1.8. *(See [8]) Let (X, S) be an S-metric space and $A \subseteq X$.*

(1) *If for every $x \in X$ there exists $r > 0$ such that $B_s(x, r) \subseteq A$, then the subset A is called an open subset of X.*
(2) *Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.*
(3) *A sequence $\{x_n\}$ in X convergent to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for each $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote by $\lim_{n\to\infty} x_n = x$.*
(4) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for each $n, m \geq 0$, $S(x_n, x_n, x_m) < \varepsilon$.*
(5) *The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.*
(6) *Let τ be the set of all $A \subseteq X$ witch $x \in A$ if and only if there exists $r > 0$ such that $B_s(x, r) \subseteq A$. Then τ is a topology on x.*

Lemma 1.9. *(See [8]) Let (X, S) be an S-metric space. If there exist the sequences $\{x_n\}$, $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

Lemma 1.10. *(See [9]) Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ for all $x, y, z \in X$. Also, $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$ by [10].*

Definition 1.11. *(See [14]) Let (X, S) and $(X^{'}, S^{'})$ be two S-metric spaces. A function $f : (X, S) \to (X^{'}, S^{'})$ is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X with $S(x_n, x_n, a) \to 0$, $S^{'}(f(x_n), f(x_n), f(a)) \to 0$. We say that f is continuous on X if f is continuous at every point $a \in X$.*

Definition 1.12. *(See [15]) Let (X, S) be an S-metric space. A pair (f, g) is said to be compatible if and only if $\lim_{n\to\infty} S(fgx_n, fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Lemma 1.13. *(See [15]) Let (X, S) be an S-metric space. If there exists two sequences x_n and y_n such that $\lim_{n\to\infty} S(x_n, x_n, y_n) = 0$, whenever x_n is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$; then $\lim_{n\to\infty} y_n = t$.*

2 Main Results

In this section, we have proved some common fixed point theorems in S- metric spaces. Let $\varPhi$ denote the class of all functions $\phi : R^+ \to R^+$ such that ϕ is nondecreasing, continuous and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$. It is clear that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence, we have $\phi(t) < t$ for all $t > 0$.

Theorem 2.1. *Let (X, S) be a complete S-metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions:*

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) U and T are continuous;
(iii) the pair (A, T) and (B, U) are compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \varPhi$ and $0 < k_1, k_2, k_3 < 1$ such that

$$\begin{aligned} S(Ax, Ay, Bz) \le \phi(max\{&S(Tx, Ty, Uz), k_1S(Ax, Ax, Tx), \\ &k_2S(Bz, Bz, Uz), k_3S(Ay, Ay, Bz)\}). \end{aligned} \tag{2}$$

Then the maps A, B, U and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary point of X. From condition (i) we can construct a sequence $\{y_n\}$ in X as follows:

$$y_{2n} = Ax_{2n} = Ux_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}, \ n \ge 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. Let $d_{n+1} = S(y_n, y_n, y_{n+1})$. Then we have

$$\begin{aligned} d_{2n+1} &= S(y_{2n}, y_{2n}, y_{2n+1}) \\ &= S(Ax_{2n}, Ax_{2n}, Bx_{2n+1}) \\ &\le \phi(max\{S(Tx_{2n}, Tx_{2n}, Ux_{2n+1}), k_1S(Ax_{2n}, Ax_{2n}, Tx_{2n}), \\ &\quad k_2S(Bx_{2n+1}, Bx_{2n+1}, Ux_{2n+1}), k_3S(Ax_{2n}, Ax_{2n}, Bx_{2n+1})\}) \\ &= \phi(max\{S(y_{2n-1}, y_{2n-1}, y_{2n}), k_1S(y_{2n}, y_{2n}, y_{2n-1}), \\ &\quad k_2S(y_{2n+1}, y_{2n+1}, y_{2n}), k_3S(y_{2n}, y_{2n}, y_{2n+1})\}) \\ &= \phi(max\{d_{2n}, k_1d_{2n}, k_2d_{2n+1}, k_3d_{2n+1}\}). \end{aligned}$$

Thus $d_{2n+1} \le \phi(d_{2n})$. By similar arguments we have,

$$\begin{aligned} d_{2n} &= S(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &= S(y_{2n}, y_{2n}, y_{2n-1}) \\ &= S(Ax_{2n}, Ax_{2n}, Bx_{2n-1}) \\ &\le \phi(max\{S(Tx_{2n}, Tx_{2n}, Ux_{2n-1}), k_1S(Ax_{2n}, Ax_{2n}, Tx_{2n}), \\ &\quad k_2S(Bx_{2n-1}, Bx_{2n-1}, Ux_{2n-1}), k_3S(Ax_{2n}, Ax_{2n}, Bx_{2n-1})\}) \\ &= \phi(max\{S(y_{2n-1}, y_{2n-1}, y_{2n-2}), k_1S(y_{2n}, y_{2n}, y_{2n-1}), \\ &\quad k_2S(y_{2n-1}, y_{2n-1}, y_{2n-2}), k_3S(y_{2n}, y_{2n}, y_{2n-1})\}) \\ &= \phi(max\{d_{2n-1}, k_1d_{2n}, k_2d_{2n-1}, k_3d_{2n}\}). \end{aligned}$$

Thus $d_{2n} \le \phi(d_{2n-1})$. Hence, for all $n \ge 2$, we have,

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &\le \phi(S(y_{n-1}, y_{n-1}, y_n)) \\ &\le \phi^2(S(y_{n-2}, y_{n-2}, y_{n-1})) \\ &\quad \cdots\cdots \\ &\le \phi^{n-1}(S(y_1, y_1, y_2)). \end{aligned}$$

By Lemma 1.10 , for $m > n$ we have

$$\begin{aligned}
S(y_n, y_n, y_m) &\le 2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_m) \\
&\le 2[S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_{n+2})] + S(y_{n+2}, y_{n+2}, y_m) \\
&\quad \cdots\cdots \\
&\le 2\sum_{i=n}^{m-2} S(y_i, y_i, y_{i+1}) + S(y_{m-1}, y_{m-1}, y_m) \\
&\le 2[\phi^{n-1}(S(y_1, y_1, y_2)) + \phi^n(S(y_1, y_1, y_2)) + \cdots + \phi^{m-2}(S(y_1, y_1, y_2))] \\
&= 2\sum_{i=n-1}^{m-2} \phi^i(S(y_1, y_1, y_2)).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, so $S(y_n, y_n, y_m) \to 0$ as $n \to \infty$. Therefore, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for each $n, m \ge n_0$, $S(y_n, y_n, y_m) < \varepsilon$. Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete S-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Ux_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = u.$$

Since T is continuous, so we have

$$\lim_{n\to\infty} T^2x_{2n+2} = Tu \;\; and \lim_{n\to\infty} TAx_{2n} = Tu.$$

And since (A, T) is compatible, then $\lim_{n\to\infty} S(ATx_{2n}, ATx_{2n}, TAx_{2n}) = 0$. So by Lemma 1.13, we have $\lim_{n\to\infty} ATx_{2n} = Tu$.

Suppose that $Tu \ne u$, by condition (2), we obtain

$$\begin{aligned}
&S(ATx_{2n}, ATx_{2n}, Bx_{2n+1}) \\
&\le \phi(max\{S(T^2x_{2n}, T^2x_{2n}, Ux_{2n+1}), k_1S(ATx_{2n}, ATx_{2n}, T^2x_{2n}), \\
&\quad k_2S(Bx_{2n+1}, Bx_{2n+1}, Ux_{2n+1}), k_3S(ATx_{2n}, ATx_{2n}, Bx_{2n+1})\}).
\end{aligned} \tag{3}$$

Taking the upper limit as $n \to \infty$ in (3), we get

$$\begin{aligned}
S(Tu, Tu, u) &\le \phi(max\{S(Tu, Tu, u), 0, 0, k_3S(Tu, Tu, u)\}) \\
&= \phi(S(Tu, Tu, u)).
\end{aligned}$$

Hence, $S(Tu, Tu, u) \le \phi(S(Tu, Tu, u)) < S(Tu, Tu, u)$, which is a contradiction. So, $Tu = u$.

Similarly, since U is continuous, we obtain that

$$\lim_{n\to\infty} U^2x_{2n+1} = Uu \;\; and \lim_{n\to\infty} UBx_{2n+1} = Uu.$$

And since (B, U) is compatible, then $\lim_{n\to\infty} S(BUx_{2n+1}, BUx_{2n+1}, UBx_{2n+1}) = 0$. So by Lemma 1.13, we have $\lim_{n\to\infty} BUx_{2n+1} = Uu$.

Suppose that $Uu \ne u$, by condition (2), we obtain

$$\begin{aligned}
&S(Ax_{2n}, Ax_{2n}, BUx_{2n+1}) \\
&\le \phi(max\{S(Tx_{2n}, Tx_{2n}, U^2x_{2n+1}), k_1S(Ax_{2n}, Ax_{2n}, Tx_{2n}), \\
&\quad k_2S(BUx_{2n+1}, BUx_{2n+1}, U^2x_{2n+1}), k_3S(Ax_{2n}, Ax_{2n}, BUx_{2n+1})\}).
\end{aligned} \tag{4}$$

Taking the upper limit as $n \to \infty$ in (4), we get

$$\begin{aligned}
S(u, u, Ru) &\le \phi(max\{S(u, u, Uu), 0, 0, k_3S(u, u, Uu)\}) \\
&= \phi(S(u, u, Uu)).
\end{aligned}$$

Consequently, $S(u,u,Uu) \leq \phi(S(u,u,Uu)) < S(u,u,Uu)$, which is a contradiction. So, $Uu = u$. Hence, we have $Tu = Uu = u$.

Also, we can apply condition (2) to obtain

$$\begin{aligned} S(Au, Au, Bx_{2n+1}) \leq \phi(max\{&S(Tu, Tu, Ux_{2n+1}), k_1 S(Au, Au, Tu), \\ &k_2 S(Bx_{2n+1}, Bx_{2n+1}, Ux_{2n+1}), k_3 S(Au, Au, Bx_{2n+1})\}). \end{aligned} \tag{5}$$

Taking the upper limit as $n \to \infty$ in (5), we have

$$\begin{aligned} S(Au, Au, u) &\leq \phi(max\{S(Tu, Tu, u), k_1 S(Au, Au, Tu), \\ &\quad k_2 S(u, u, u), k_3 S(Au, Au, u)\}) \\ &\leq max\{k_1, k_3\} S(Au, Au, u). \end{aligned}$$

if $Au \neq u$, then this implies that $max\{k_1, k_3\} \geq 1$, which is a contradiction. Hence, from $\phi(t) < t$ for all $t > 0$, we have $Au = u$.

Finally, by using of condition (2), we get

$$\begin{aligned} S(u, u, Bu) &= S(Au, Au, Bu) \\ &\leq \phi(max\{S(Tu, Tu, Uu), k_1 S(Au, Au, Tu), \\ &\quad k_2 S(Bu, Bu, Uu), k_3 S(Au, Au, Bu)\}) \\ &\leq max\{k_2, k_3\} S(u, u, Bu). \end{aligned}$$

if $Bu \neq u$, then this implies that $max\{k_2, k_3\} \geq 1$, which is a contradiction. Hence, from $\phi(t) < t$ for all $t > 0$, we have $Bu = u$.

Thus, we have $Tu = Uu = Au = Bu = u$, that is, u is a common fixed point of A, B, U and T.

Suppose that p is another common fixed point of A, B, U and T, that is, $p = Ap = Bp = Up = Tp$. If $u \neq p$, then by condition (2), we have that

$$\begin{aligned} S(u, u, p) &= S(Au, Au, Bp) \\ &\leq \phi(max\{S(Tu, Tu, Up), k_1 S(Au, Au, Tu), \\ &\quad k_2 S(Bp, Bp, Up), k_3 S(Au, Au, Bp)\}) \\ &= \phi(max\{S(u, u, p), k_1 S(u, u, u), k_2 S(p, p, p), k_3 S(u, u, p)\}) \\ &\leq \phi(S(u, u, p)). \end{aligned}$$

Hence, $S(u,u,p) \leq \phi(S(u,u,p)) < S(u,u,p)$, which is a contradiction. Hence, $u = p$. Therefore, u is a unique common fixed point of A, B, U and T. This completes the proof. □

Remark 2.2. *Theorem 2.1 of this paper extends Theorem 2.1 of [14] from two mappings to four mappings and changes condition from closed subset to continuous and strengthens condition from weakly compatible to compatible. And it's worth noting that Theorem 2.1 of this article is a further extension of Theorem 2.2 of [15].*

Remark 2.3. *if $x = y$ in (2) of Theorem 2.1, and by (1), then we have*

$$S(Ay, Ay, Bz) = d(Ay, Bz) \leq \phi(max\{d(Ty, Uz), k_1 d(, Ay, Ty), k_2 d(Bz, Uz), k_3 d(Ay, Bz)\}).$$

Corollary 2.4. *Let (X, S) be a complete S-metric space and let $A, U : X \to X$ be two mappings such that*

(i) $A(X) \subseteq U(X)$;
(ii) U is continuous;
(iii) the pair (A, U) is compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2, k_3 < 1$ such that

$$\begin{aligned} S(Ax, Ay, Az) \leq \phi(max\{&S(Ux, Uy, Uz), k_1 S(Ax, Ax, Ux), \\ &k_2 S(Az, Az, Uz), k_3 S(Ay, Ay, Az)\}). \end{aligned}$$

Then A and U have a unique common fixed point in X.

Proof. If we take $A = B$ and $U = T$ in Theorem 2.1, then Theorem 2.1 follows that A and U have a unique common fixed point. □

Corollary 2.5. *Let (X, S) be a complete S-metric space and let $A, B, : X \to X$ be two mappings such that*

$$S(Ax, Ay, Bz) \leq \phi(max\{S(x, y, Uz), k_1 S(Ax, Ax, x), k_2 S(Bz, Bz, z), k_3 S(Ay, Ay, Bz)\}).$$

Then A and B have a unique common fixed point in X.

Proof. If we take U and T as identity mappings on X, then Theorem 2.1 follows that A and B have a unique common fixed point. □

Theorem 2.6. *Let (X, S) be a complete S-metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions:*

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) U and T are continuous;
(iii) the pair (A, T) and (B, U) are compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2, k_3 < 1$ and $p, q \in N$ such that

$$\begin{aligned} S(A^p x, A^p y, B^q z) \leq \phi(max\{&S(Tx, Ty, Uz), k_1 S(A^p x, A^p x, Tx), \\ &k_2 S(B^q z, B^q z, Uz), k_3 S(A^p y, A^p y, B^q z)\}). \end{aligned} \tag{6}$$

Then the maps A, B, U and T have a unique common fixed point.

Proof. (i) When $p = q = 1$, we have $Au = Bu = Uu = Tu = u$, u is a unique common fixed point of A, B, U and T.

(ii) If one of p and q is not equal to 1. Similar to Theorem 2.1, we can prove that A^p, B^q, U and T have a unique common fixed point u, that is, $A^p u = B^q u = Uu = Tu = u$. Now, we should prove u is unique common fixed point of A, B, U and T. Indeed,

$$A^p(Au) = A^{p+1}u = A(A^p u) = Au = A(Tu) = T(Au).$$

So, Au is a common fixed point of A^p and T. Suppose that $Au \neq u$, and

$$\begin{aligned} S(Au, Au, u) &= S(Au, Au, B^q u) \\ &= S(A^p(Au), A^p(Au), B^q u) \\ &\leq \phi(max\{S(T(Au), T(Au), Uu), k_1 S(A^p(Au), A^p(Au), T(Au)), \\ &\quad k_2 S(B^q u, B^q u, Uu), k_3 S(A^p(Au), A^p(Au), B^q u)\}) \\ &= \phi(max\{S(Au, Au, u), k_1 S(Au, Au, Au), \\ &\quad k_2 S(u, u, u), k_3 S(Au, Au, u)\}) \\ &\leq \phi(S(Au, Au, u)). \end{aligned}$$

Hence, $S(Au, Au, u) \leq \phi(S(Au, Au, u)) < S(Au, Au, u)$, which is a contradiction. It means that $Au = u$. And,

$$B^q(Bu) = B^{q+1}u = B(B^q u) = Bu = B(Uu) = U(Bu).$$

Thus, Bu is a common fixed point of B^q and U. Suppose that $Bu \neq u$, and

$$\begin{aligned} S(u, u, Bu) &= S(A^p u, A^p u, B^q(Bu)) \\ &\leq \phi(max\{S(Tu, Tu, U(Bu)), k_1 S(A^p u, A^p u, Tu), \\ &\quad k_2 S(B^q(Bu), B^q(Bu), U(Bu)), k_3 S(A^p u, A^p u, B^q(Bu))\}) \\ &= \phi(max\{S(u, u, Bu), k_1 S(u, u, u), \\ &\quad k_2 S(Bu, Bu, Bu), k_3 S(u, u, Bu)\}) \\ &\leq \phi(S(u, u, Bu)). \end{aligned}$$

Hence, $S(u,u,Bu) \leq \phi(S(u,u,Bu)) < S(u,u,Bu)$, which is a contradiction. It means that $Bu = u$. Therefore, $Au = Bu = Uu = Tu = u$. Thus, u is a unique common fixed point of A, B, U and T. This completes the proof. □

Remark 2.7. *if $x = y$ in (6) of Theorem 2.6, and by (1), then we have*

$$S(A^py, A^py, B^qz) = d(A^py, B^qz) \leq \phi(max\{d(Ty, Uz), k_1d(, A^py, Ty), \\ k_2d(B^qz, Uz), k_3d(A^py, B^qz)\}).$$

Corollary 2.8. *Let (X, S) be a complete S-metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions:*
(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) U and T are continuous;
(iii) the pair (A, T) and (B, U) are compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2, k_3 < 1$ and $p \in N$ such that

$$S(A^px, A^py, B^pz) \leq \phi(max\{S(Tx, Ty, Uz), k_1S(A^px, A^px, Tx), \\ k_2S(B^pz, B^pz, Uz), k_3S(A^py, A^py, B^pz)\}).$$

Then the maps A, B, U and T have a unique common fixed point.

Proof. Let $p = q$ and the process of proof is similar to the proof of Theorem 2.6. □

Theorem 2.9. *Let (X, S) be a complete S-metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions:*
(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) U and T are continuous;
(iii) the pair (A, T) and (B, U) are compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2 < 1$ such that

$$S(Ax, Ay, Bz) \leq \phi(max\{S(Tx, Ty, Uz), k_1S(Ax, Tx, Tx), k_2S(Bz, Uz, Uz)\})$$

Then the maps A, B, U and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary point of X. From condition (i) we can construct a sequence $\{y_n\}$ in X as follows:

$$y_{2n} = Ax_{2n} = Ux_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}, \ n \geq 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. Let $d_{n+1} = S(y_n, y_n, y_{n+1})$. Then we have

$$\begin{aligned} d_{2n+1} &= S(y_{2n}, y_{2n}, y_{2n+1}) \\ &= S(Ax_{2n}, Ax_{2n}, Bx_{2n+1}) \\ &\leq \phi(max\{S(Tx_{2n}, Tx_{2n}, Ux_{2n+1}), k_1S(Ax_{2n}, Tx_{2n}, Tx_{2n}), \\ &\quad k_2S(Bx_{2n+1}, Ux_{2n+1}, Ux_{2n+1})\}) \\ &= \phi(max\{S(y_{2n-1}, y_{2n-1}, y_{2n}), k_1S(y_{2n}, y_{2n-1}, y_{2n-1}), \\ &\quad k_2S(y_{2n+1}, y_{2n}, y_{2n})\}) \\ &= \phi(max\{d_{2n}, k_1d_{2n}, k_2d_{2n+1}\}). \end{aligned}$$

Thus $d_{2n+1} \leq \phi(d_{2n})$. By similar arguments we have,

$$\begin{aligned} d_{2n} &= S(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &= S(y_{2n}, y_{2n}, y_{2n-1}) \\ &= S(Ax_{2n}, Ax_{2n}, Bx_{2n-1}) \\ &\leq \phi(max\{S(Tx_{2n}, Tx_{2n}, Ux_{2n-1}), k_1S(Ax_{2n}, Tx_{2n}, Tx_{2n}), \\ &\quad k_2S(Bx_{2n-1}, Ux_{2n-1}, Ux_{2n-1})\}) \\ &= \phi(max\{S(y_{2n-1}, y_{2n-1}, y_{2n-2}), k_1S(y_{2n}, y_{2n-1}, y_{2n-1}), \\ &\quad k_2S(y_{2n-1}, y_{2n-2}, y_{2n-2})\}) \\ &= \phi(max\{d_{2n-1}, k_1d_{2n}, k_2d_{2n-1}\}). \end{aligned}$$

Thus $d_{2n} \leq \phi(d_{2n-1})$.

The process of next proof is similar to the proof of Theorem 2.1. □

Theorem 2.10. *Let (X, S) be a complete S-metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions:*

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) U and T are continuous;
(iii) the pair (A, T) and (B, U) are compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2 < 1$ and $p, q \in N$ such that

$$S(A^p x, A^p y, B^q z) \leq \phi(max\{S(Tx, Ty, Uz), k_1 S(A^p x, Tx, Tx), k_2 S(B^q z, Uz, Uz)\})$$

Then the maps A, B, U and T have a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 2.6. □

Corollary 2.11. *Let (X, S) be a complete S-metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions:*

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) U and T are continuous;
(iii) the pair (A, T) and (B, U) are compatible;
(iv) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2 < 1$ and $p \in N$ such that

$$S(A^p x, A^p y, B^p z) \leq \phi(max\{S(Tx, Ty, Uz), k_1 S(A^p x, Tx, Tx), k_2 S(B^p z, Uz, Uz)\})$$

Then the maps A, B, U and T have a unique common fixed point.

Example 2.12. *Let $X = [0, 1]$ and (X, S) be a complete S-metric space. For any $x, y, z \in X$, define $S(x, y, z) = |x - z| + |y - z|$ and mappings $A, B, U, T : X \to X$ on X by*

$$Ax = \frac{x}{16},\ Bx = \frac{x}{8},\ Tx = \frac{x}{4},\ Ux = \frac{x}{2}.$$

Then, it is easy to see that $A(X) \subseteq U(X)$ and $B(X) \subseteq T(X)$. Moreover, the pair (A, T) and (B, U) are compatible mappings.

Also, for all $x, y, z \in X$, we have

$$\begin{aligned} S(Ax, Ay, Bz) &= |Ax - Bz| + |Ay - Bz| \\ &= |\frac{x}{16} - \frac{z}{8}| + |\frac{y}{16} - \frac{z}{8}| \\ &= \frac{1}{4}|\frac{x}{4} - \frac{z}{2}| + \frac{1}{4}|\frac{y}{4} - \frac{z}{2}| \\ &\leq \frac{3}{4}|Tx - Rz| + \frac{3}{4}|Ty - Rz| \\ &\leq S(Tx, Ty, Uz) \\ &\leq \phi(max\{S(Tx, Ty, Uz), k_1 S(Ax, Ax, Tx), \\ &\quad k_2 S(Bz, Bz, Uz), k_3 S(Ay, Ay, Bz)\}), \end{aligned}$$

where $0 < k_1, k_2, k_3 < 1$. Therefore, all the conditions of Theorem 2.1 are satisfied and 0 is the unique common fixed point of A, B, U and T.

3 Conclusion

In this paper, we established some common fixed point theorems about two pair maps in S-metric space. The presented theorems extend, generalize and improve many existing results in S-metric spaces in the literature. Our results differ from those in the literature and they may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

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References

1. Banach, S.: Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fund. Math. 3(1), 133-181 (1922).
2. Kadelburg, Z., Murthy, P.P., Radenović, S.: Common fixed points for expansive mappings in cone metric spaces. Int. J. Math. Anal. 5(27), 1309-1319 (2011).
3. Aghajani, A., Abbas, M., Roshan, J.R.: Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math. Slovaca. 64(4), 941-960 (2014).
4. Shatanawi, W.: Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces. Fixed Point Theory Appl. (2010), Article ID 181650.
5. Sedghi, S., Shobe, N., Aliouche, A.: A generalization of fixed point theorems in S-metric spaces. Mat. Vesnik. 64(3), 258-266 (2012).
6. Bakhtin I.A.: The contraction principle in quasimetric spaces. Func. An., Ulianowsk, Gos. Ped. Ins. 30, 26-37 (1989).
7. Sedghi, S., Dung, N. V.: Fixed point theorems on S-metric spaces. Mat. Vesnik. 66(1), 113-124 (2014).
8. Sedghi, S., Shobe, N., Došenović T.: Fixed point results in S-metric spaces. Nonlinear Funct. Anal. Appl. 20(1), 55-67 (2015).
9. Dung N.V.: On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces. Fixed Point Theory Appl. 2013,48 (2013).
10. Sedghi, S., Altun, I., Shobe, N., Salahshour, M. A.: Some properties of S-metric spaces and fixed point results. Kyungpook Math. J. 54(1), 113-122 (2014).
11. Sedghi, S., Došenović, T., Mahdi Rezaee, M., Radenovicąä, S.: Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces. Acta Univ. Sapientiae Math. 8(2), 298-311 (2016).
12. Mojaradi Afra, J.: Double contraction in S-metric spaces. Int.J. Math. Anal. 9(3), 117-125 (2015).
13. Prudhvi, K.: Fixed point theorems in S-metric spaces. Univers. J. Comput. Math. 3(2), 19-21 (2015).
14. Kyu Kim, J., Sedghi, S., Gholidahneh, A., Mahdi Rezaee, M.: Fixed point theorems in S-metric spaces. East Asian Math. J. 32(5), 677-684 (2016).
15. Sedghi, S., Shobe, N., Došenović , T.: Common fixed point of four maps in S-metric spaces. Mathematical Sciences. 12: 137-143 (2018).
16. Sedghi, S., Gholidahneh, A., Došenović, T., Esfahani, J., Radenovic ąä, S.: Common fixed point of four maps in S_b-metric spaces. J. Linear Topol. Algebra 05(02), 93-104 (2016).