

# Quantum Mechanics of ‘Two Path’ Experiments: Comparison of Neutrino Oscillations and the Young Double Slit

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**Abstract** Feynman path integral analyses of a two-neutrino-flavour electron appearance experiment and the Young double slit are compared. The comparison reveals a conceptual flaw in previous work of the present author that led to a false claim of the incorrectness of standard formulas for the oscillation phase. In both calculations, path amplitudes add coherently, but no putative ‘neutrino flavour eigenstates’ are invoked in the former case. It is shown that the coherent production of these eigenstates is incompatible with the measured values of  $\Gamma(\pi \rightarrow e\nu)/\Gamma(\pi \rightarrow \mu\nu)$  and the PMNS matrix elements. Applications of the path integral approach to other two-path quantum interference experiments are compared with neutrino oscillations, and other treatments of the latter in the literature are critically discussed.

**Keywords:** Quantum mechanics, neutrino oscillations.

## 1 Introduction

The fundamental theoretical construct of Feynman’s path integral formulation of quantum mechanics [1,2,3] is the probability amplitude corresponding to a definite sequence of space-time events. The phase of this amplitude is the sum of the phases of the different sub-amplitudes whose product constitutes the probability amplitude for a given sequence of space-time events. The probability amplitudes: ‘path amplitudes’ for two such sequences interfere if, and only if, the initial and final states of the two amplitudes are the same. This implies for the case of physical optics, as explained in a popular book by Feynman [4], that the photon in the different path amplitudes for, say, the Young double slit experiment, must be created at different times for paths of different lengths in order for quantum mechanical interference effects to occur. The phase of the probability amplitude —of crucial importance in the determination of interference effects— was likened by Feynman in Ref. [4] to the position of the hand of an ‘imaginary stopwatch’. Concerning the motion of this hand, Feynman stated:

"The rate of turning depends on the color of the light: the amplitude for a blue source turns nearly twice as fast as for a red source... So the timer we used for the ‘imaginary stopwatch’ was the monochromatic source: —in reality the angle of the amplitude for a given path depends on what *time* the photon is emitted from the source." (Feynman’s italics)

As explained below, there is a close analogy between a photonic double slit experiment and a two-flavour neutrino oscillation experiment. In the former case the phase difference governing interference effects arises from different path lengths and different production times, in the latter, partially from different masses appearing in the space-time neutrino propagators, partially from different production times, for equal path lengths. In both cases it is essential that the particles (a photon in one case, neutrinos of different mass in the other) are *created at different times* in the interfering amplitudes. This is necessary in order to satisfy the space-time geometric constraint:  $s = vt$  where  $s$ ,  $v$  and  $t$  are, respectively, the path length, the particle velocity and the time-of-flight of the particle. As shown below, this requirement is in contradiction with the ansatz of Eq. (5.1) below that is conventionally assumed in order to derive the phase of neutrino oscillations in vacuo. The experimental validity of the path integral approach has been demonstrated in a variety of different experiments, some of which are briefly described in Section 7 of the present paper. There is no reason to suppose that this approach is inapplicable to the neutrino oscillation problem.

The present author previously published related work [11,12,2] somewhat more than a decade ago. A summary of the ensuing controversies and references to other related literature can be found in Refs. [13,14]. The novel features of the present article are: the comparison of neutrino oscillations and a Young double slit experiment, that is more than just an interesting pedagogical exercise, because it reveals an important shortcoming in my previous analysis of the former problem necessitating the new analysis which is presented in of Section 4 below. Also new is the work presented in Section 6 showing the incompatibility of the hypothesis of the coherent production of a ‘neutrino flavour eigenstate’ at a unique time with measured pion branching ratios. The brief discussion of the relation of quantum field theory to neutrino oscillations in Section 8 is also given here for the first time.

This paper is organised as follows: in the following section neutrino mass and flavour eigenstates are defined. In Section 3 the probability amplitude analyses of a Young double slit experiment with photons and a two flavour neutrino oscillation experiment are compared. Section 4 contains a detailed probability amplitude analysis of two-flavour neutrino oscillations, while the standard ‘plane wave’ analysis is recalled in Section 5. In Section 6 the role of mass eigenstates and flavour eigenstates in pion decay is compared and it is shown the the experimental ratio:  $\Gamma(\pi^+ \rightarrow e^+\nu)/\Gamma(\pi^+ \rightarrow \mu^+\nu)$  is incompatible with prompt production of neutrino flavour eigenstates in pion decay as assumed to be the case in the standard ‘plane wave’ analysis of neutrino oscillations. In Section 7 it is shown how the basic hypotheses of the probability amplitude analysis of neutrino oscillations are confirmed by successful analyses of related experiments. Section 8 contains a brief discussion of the role of quantum field theory in the description of neutrino oscillations, while conclusions are given in Section 9.

## 2 Mass and Flavour Neutrino Eigenstates

In the Standard Electroweak Model (SEM), the coupling of a charged lepton:  $\ell_i$ , of generation  $i$  ( $i = e, \mu, \tau$ ) and a neutrino mass eigenstate:  $\nu_j$ , ( $j = 1, 2, 3$ ), to the W-boson, is proportional to  $ij$ th component of the leptonic charged current [5]:

$$J_\mu(CC)^{lept} = \sum_{i,j} \bar{\psi}_{\ell_i} \gamma_\mu (1 - \gamma_5) U_{ij} \psi_{\nu_j} = \sum_i \bar{\psi}_{\ell_i} \gamma_\mu (1 - \gamma_5) \psi_{\nu_i} \quad (2.1)$$

where

$$\psi_{\nu_i} \equiv \sum_j U_{ij} \psi_{\nu_j}$$

and  $U_{ij}$  is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) [6,7] charged-lepton-flavour/neutrino-mass mixing matrix. Table 1 shows the elements of this matrix obtained from experimental measurements of solar, atmospheric and reactor neutrino oscillations. The non-diagonal nature of this matrix gives evidence for strong violation of generation number (or lepton flavour) by  $J_\mu(CC)^{lept}$ . Conservation of generation number corresponds to a diagonal PMNS matrix with  $\nu_1 = \nu_e$ ,  $\nu_2 = \nu_\mu$  and  $\nu_3 = \nu_\tau$ . This is the conventional massless neutrino scenario. With massive neutrinos and a non-diagonal PMNS matrix the leptonic charged current (2.1) may be formally written either in terms of neutrino mass eigenstates:  $\psi_{\nu_j}$  ( $j = 1, 2, 3$ ) or of flavour eigenstates:  $\psi_{\nu_i}$  ( $i = e, \mu, \tau$ ). However in all *physical processes* only mass eigenstates are produced, as a consequence of energy-momentum conservation, so all physical predictions are derived from the current given in the first member of (2.1).

An analysis of two-flavour neutrino oscillations, following pion decay at rest, within Feynman’s path integral formulation of quantum mechanics, is presented below. In this case, without loss of generality, the PMNS matrix elements are assumed to be real numbers.

There is also no loss of generality, for the questions of physical principle discussed in this paper, due to the restriction to two-flavour oscillations rather than the three-flavour oscillations observed in the real world. This is due to the circumstance that the different contributions to the detection probability of a particular charged lepton, of the interference of the probability amplitudes corresponding to different mass eigenstates have an identical structure for each pair of interfering amplitudes. Thus the interference term that gives the oscillation effect for three flavours is the sum of three similar two-flavour oscillation terms. In the quantitative analysis of pion decay in different models considered in Section 6 below three-flavour mixing is taken fully into account.

**Table 1.** Values of the PMNS charged lepton flavour/neutrino mass mixing matrix  $U_{ij}$  as derived from solar, atmospheric and reactor neutrino oscillation data. The matrix elements are calculated in terms of two-flavour mixing angles using the three-flavour parameterisation of the PMNS matrix [8] with  $\sin \theta_{12} = 0.558 + 0.015 - 0.014$  [9],  $\sin \theta_{23} = 0.648 + 0.059 - 0.024$  [9],  $\sin \theta_{13} = 0.154 + 0.011 - 0.016$  [10] and  $\delta = 0$ . Average uncertainties are used to calculate the matrix elements.

$j$	1 ( $\nu_1$ )	2 ( $\nu_2$ )	3 ( $\nu_3$ )
$i$			
1 ( $e$ )	$0.82 \pm 0.01$	$0.551 \pm 0.015$	$0.154 \pm 0.014$
2 ( $\mu$ )	$-0.508 \pm 0.029$	$0.574 \pm 0.032$	$0.640 \pm 0.042$
3 ( $\tau$ )	$0.265 \pm 0.025$	$-0.603 \pm 0.035$	$0.753 \pm 0.038$

### 3 Comparison of Young Double Slit and Neutrino Oscillation Experiments

A comparison of two-flavour neutrino oscillations with an analogous two-path experiment, the Young double slit, in physical optics, is now made. This comparison will reveal an incorrect physical postulate in previous treatments [11,12,2] of neutrino oscillations by the present author. Following the sequential factorisation law [1,2,3] for constructing path amplitudes, each such amplitude in a two-flavour neutrino oscillation experiment or a two-path experiment in photonic physical optics, will be the product of the following amplitudes:

- (i) The amplitude to produce the source particle.
- (ii) The decay amplitude of the source particle into a final state containing a neutrino mass eigenstate or a photon.
- (iii) The space-time propagator of the neutrino or the photon.
- (iv) The amplitude of the process by which the neutrino or photon is detected.

The superposition principle for path amplitudes [1,2,3] requires that if, and only if, the path amplitudes have the same initial and final states they must be added coherently, i.e. the amplitudes, not the modulus squared of the amplitudes, must be summed. This coherence condition is completely different to the hypothesis to be discussed below, that is the basis of ‘standard’ neutrino oscillation phenomenology that a ‘neutrino flavour eigenstate’ that is a superposition of neutrino mass eigenstates, is produced at some fixed time. The production amplitude in (i) is common to both path amplitudes and therefore contributes only an overall multiplicative factor to the oscillation probability or interference pattern. For the neutrino oscillation experiment, the initial state of the path amplitudes is that of the pion at the instant of its creation. The amplitude in (ii) is a function of the time interval,  $t_P$ , after the source particle is created, at which the decay occurs [15,2,3]:

$$\langle f|i \rangle_{t_P} = \exp \left[ -i \frac{(E_i - E_f)t_P}{\hbar} \right] \langle f|i \rangle_0 \quad (3.1)$$

where  $\langle f|i \rangle_0$  and  $\langle f|i \rangle_{t_P}$  are the transition amplitudes at time zero and  $t_P$  respectively. The phase  $(E_i - E_f)t_P/\hbar$ , proportional to  $t_P$ , is the angle of rotation of Feynman’s ‘imaginary stopwatch’ hand. The suffix ‘P’ stands for ‘Production’ (of the neutrino or photon). In the formula (3.1) it is assumed that the lifetime of the source particle is much greater than the difference between the times-of-flight of the neutrinos or photons in the two paths. The source particle is produced at time zero in both path amplitudes. In the physical optics application of (3.1)  $E_i$  and  $E_f$  are the energies of atomic states and  $E_i - E_f = E_\gamma$  where  $E_\gamma$  is the photon energy. The conceptual error in [11,12,2] and earlier versions of the present paper [16] was to replace the amplitude (ii) by the space-time propagator of the source particle. Since the same laws of physics should apply for both neutrino oscillations and physical optics this corresponds, in the latter case, to replacing  $E_\gamma$  by  $M_i c^2$  where  $M_i$  is the mass of the unstable source atom! The ‘photon wavelength’ governing interference effects would then be smaller by the factor  $E_\gamma/(M_i c^2)$  as compared with the value in the classical wave theory of light [2,3] — evidently at variance with experiment. Indeed, the exact correspondence of the general formula (3.1) with the corresponding one for neutrino production in  $\beta$ -decay, Eq. (73) of Ref. [11], should have made obvious the incorrect nature of Eq. (72) in

the same article, used for pion decay, and leading to a non-standard prediction of the neutrino oscillation phase.

Since the space-time propagator of a free particle has the phase:  $(rp - Et)/\hbar$  [2], and for a photon  $c = r/t = E/p$ , the propagator phase vanishes [4] so that the path amplitude phase resides entirely in (ii) and is given by (3.1) with  $E_i - E_f = E_\gamma$ . For the case of neutrino oscillations (3.1) holds with  $E_i - E_f = E_{\nu_j} \equiv E_j$ , whereas the phase of the neutrino propagator is [11,2]:  $-m_j c^2 \tau_F / \hbar$  where  $\tau_F$  is the time-of-flight of the neutrino in its rest frame.

The final state of both path amplitudes is that of the detection process described by the amplitude (iv).

For more details of the Feynman path analysis of the Young double slit experiment for both photons and massive particles see Ref. [2]. A similar description of reflection diffraction grating experiments is given in Ref. [3].

## 4 Probability Amplitude Analysis of Two-Flavour Neutrino Oscillations

Consider now production of the neutrino mass eigenstates  $\nu_1$  or  $\nu_2$  in the two-body decay at rest of a positively charged pion:  $\pi^+ \rightarrow \mu^+ \nu_1$  or  $\mu^+ \nu_2$ . A ‘neutrino oscillation’ effect is manifested by detection of a neutrino via the processes:  $(\nu_1, \nu_2) n \rightarrow e^- p$  at a fixed distance,  $L$ , from the source. In units with  $\hbar = c = 1$  the path amplitude for the mass eigenstate  $\nu_j$  is, up to a overall multiplicative constant [11,2]:

$$A_{e\mu\pi}^j(t_P^j) = U_{ej} \langle e^- | \nu \rangle \exp \left[ -i \frac{m_j^2 L}{p_j} \right] \exp \left\{ -i E_j t_P^j \right\} U_{\mu j} \langle \nu \mu^+ | \pi^+ \rangle \quad (4.1)$$

where the ‘reduced’ decay and scattering amplitudes  $\langle \nu \mu^+ | \pi^+ \rangle$  and  $\langle e^- | \nu \rangle$  are defined according to the relations <sup>1</sup>:

$$\langle \nu_j \mu^+ | \pi^+ \rangle \equiv U_{\mu j} \langle \nu \mu^+ | \pi^+ \rangle, \quad \langle e^- | \nu_j \rangle \equiv U_{ej} \langle e^- | \nu \rangle. \quad (4.2)$$

The amplitudes (ii)-(iv) are written as factors from right to left on the right side of (3). Extracting the the modulus and phase of the path amplitude in (4.1):

$$A_{e\mu\pi}^j(t_P^j) = U_{ej} U_{\mu j} |\langle e^- | \nu \rangle| |\langle \nu \mu^+ | \pi^+ \rangle| e^{i(\phi_j + \phi_0)} \equiv A_{e\mu\pi}^{0j} e^{i(\phi_j + \phi_0)} \quad (4.3)$$

where  $\phi_0$  is a possible flavour-independent phase associated with the amplitudes  $\langle \nu \mu^+ | \pi^+ \rangle$  and  $\langle e^- | \nu \rangle$  and

$$\phi_j \equiv - \left( \frac{m_j^2 L}{p_j} + E_j t_P^j \right). \quad (4.4)$$

Quantum mechanical superposition of the path amplitudes [1,2,3] gives, for the probability to detect an electron:

$$P_{e\mu\pi} = |A_{e\mu\pi}^1 + A_{e\mu\pi}^2|^2 = (A_{e\mu\pi}^{01})^2 + (A_{e\mu\pi}^{02})^2 + 2A_{e\mu\pi}^{01} A_{e\mu\pi}^{02} \cos(\phi_1 - \phi_2). \quad (4.5)$$

Introducing the neutrino production time difference:  $2\Delta t_P$  and the mean neutrino production time  $\bar{t}_P$ :

$$\Delta t_P \equiv \frac{t_P^1 - t_P^2}{2}, \quad \bar{t}_P \equiv \frac{t_P^1 + t_P^2}{2} \quad (4.6)$$

enables the phase difference between the two path amplitudes to be written, using (4.4), as:

$$\Delta\phi_{12} \equiv \phi_1 - \phi_2 = \left( \frac{m_2^2}{p_2} - \frac{m_1^2}{p_1} \right) L - (E_1 + E_2)\Delta t_P + (E_2 - E_1)\bar{t}_P. \quad (4.7)$$

If  $t_f^1, t_f^2$  are the times-of-flight of  $\nu_1$  and  $\nu_2$  between production and detection at the common time  $t_D$  then

$$t_D = t_P^1 + t_f^1 = t_P^2 + t_f^2 \quad (4.8)$$

<sup>1</sup> The Dirac ‘bra’ and ‘ket’ notation is used to label invariant amplitudes in Eq. (4.1). The amplitude  $\langle \nu_j \mu^+ | \pi^+ \rangle$  is equal to  $\mathcal{M}_{\bar{\ell}\nu_j}$  defined in Eq. (6.8), with  $\bar{\ell} = \mu^+$ .

so that

$$t_{\text{P}}^1 - t_{\text{P}}^2 = 2\Delta t_{\text{P}} = t_f^2 - t_f^1 \quad (4.9)$$

and since

$$t_f^j = \frac{L}{v_j} = \frac{E_j L}{p_j} \quad j = 1, 2 \quad (4.10)$$

it follows that

$$\Delta t_{\text{P}} = \frac{L}{2} \left( \frac{E_2}{p_2} - \frac{E_1}{p_1} \right). \quad (4.11)$$

Exact relativistic two-body kinematics of the decay process  $\pi \rightarrow \mu\nu_j$ <sup>2</sup> gives:

$$E_j = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} + \frac{m_j^2}{2m_\pi} \equiv E_\nu + \frac{m_j^2}{2m_\pi} \quad j = 1, 2, \quad (4.12)$$

$$E_2 - E_1 = \frac{\Delta m_{21}^2}{2m_\pi}, \quad \Delta m_{21}^2 \equiv m_2^2 - m_1^2. \quad (4.13)$$

Since

$$p_j = E_j - \frac{m_j^2}{2E_\nu} + \mathcal{O}(m_j^4) \quad (4.14)$$

(4.11) gives

$$\Delta t_{\text{P}} = \frac{\Delta m_{21}^2 L}{4E_\nu^2} + \mathcal{O}(m_j^4) \quad (4.15)$$

Combining (4.11)-(4.15) and (4.7) gives<sup>3</sup>:

$$\Delta\phi_{12} = \frac{\Delta m_{21}^2}{2E_\nu} \left[ L + \frac{E_\nu c \bar{t}_{\text{P}}}{E_\pi} \right] + \mathcal{O}(m_j^4) \quad (4.16)$$

where both terms in the square bracket are of dimension [L]. Inserting the values of the PMNS matrix elements in terms of the two-flavour mixing angle  $\theta_{12}$ :

$$\begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} \\ -\sin \theta_{12} & \cos \theta_{12} \end{pmatrix} \quad (4.17)$$

in (4.5) gives

$$P_{e\mu\pi} = (A_{e\mu\pi}^0)^2 2 \cos^2 \theta_{12} \sin^2 \theta_{12} (1 - \cos \Delta\phi_{12}) \quad (4.18)$$

where

$$A_{e\mu\pi}^0 \equiv |\langle e^- | \nu \rangle| |\langle \nu \mu^+ | \pi^+ \rangle|. \quad (4.19)$$

The maximum electron production rate occurs for  $\Delta\phi_{12} \simeq \pi$ , which, inserting the measured value [17] of  $\Delta m_{21}^2 = 7.58 \times 10^{-5} \text{ (eV)}^2$  as well as  $E_\nu = 29.8 \text{ MeV}$  requires that  $L + E_\nu c \bar{t}_{\text{P}} / E_\pi = 490 \text{ km}$ . Since  $c \bar{t}_{\text{P}} \simeq c \tau_\pi = 7.8 \text{ m}$ , the term in (4.16) containing the mean production time  $\bar{t}_{\text{P}}$  may be neglected for experimentally interesting values of  $\Delta\phi_{12}$ . Eq. (4.18) may then be written as:

$$P_{e\mu\pi} = (A_{e\mu\pi}^0)^2 \sin^2 2\theta_{12} \sin^2 \frac{\Delta m_{21}^2 L}{4E_\nu}. \quad (4.20)$$

<sup>2</sup> Actually, due to the unstable nature of the source pion and the decay muon, the masses of these particles are not constant but vary according to Breit-Wigner distributions with widths  $\Gamma_\pi$ ,  $\Gamma_\mu$  respectively. A detailed discussion of the associated damping effects, which are found to be negligible as compared to any actual or foreseeable experimental uncertainties, may be found in Ref. [12], together with calculations of Doppler shifts due to thermal motion of the stopped pion. A critical discussion of the *ad hoc* Gaussian wave-packets introduced by some authors into discussions of neutrino oscillations, is in Ref. [11].

<sup>3</sup> For clarity in the discussion of the physical values of the parameters in (4.16) the speed of light,  $c$ , is written explicitly in the last term of this equation.

Thus, contrary to assertions in previous papers [11,12,2] by the present author, correct application of the Feynman path integral formulation reproduces the standard formula  $\Delta\phi_{12} = \Delta m_{21}^2 L / (2E_\nu)$  for the ‘vacuum oscillation’ phase difference.

The above calculation shows that there are two distinct contributions to the phase difference  $\Delta\phi_{12}$  at leading order in the neutrino masses. The first, originating in the neutrino propagator, is the  $L$ -dependent term in (4.7) that gives the contribution:

$$\Delta\phi_{12}^\nu \equiv \left( \frac{m_2^2}{p_2} - \frac{m_1^2}{p_1} \right) L = \frac{\Delta m_{21}^2 L}{E_\nu} + \mathcal{O}(m_j^4). \quad (4.21)$$

The second, originating in the decay amplitude of the source pion is the  $\Delta t_P$  dependent term in (4.7):

$$\Delta\phi_{12}^\pi \equiv -(E_1 + E_2)\Delta t_P = -\frac{\Delta m_{21}^2 L}{2E_\nu} + \mathcal{O}(m_j^4). \quad (4.22)$$

The phase  $\Delta\phi_{12}^\nu$  above, associated with neutrino propagation, was correctly given [11] in the seminal paper of Gribov and Pontecorvo [18] on neutrino oscillations.

## 5 Conventional ‘Plane Wave’ Analysis of Neutrino Oscillations

How the factor two difference between  $\Delta\phi_{12}^\nu$  given in Eq. (4.21) and the overall interference phase  $\Delta\phi_{12}$  of Eq. (4.16) is obtained in the conventional ‘plane wave’ derivation of the latter phase difference, without any consideration of the contribution from the source particle decay amplitude, will now be explained [11]. A typical such derivation is to be found in the review article of Kayser in the 2008 ‘Review of Particle Properties’ [19]. There the interference phase difference is asserted to be:

$$\Delta\tilde{\phi}_{12}^\nu = (p_1 - p_2)L - (E_1 - E_2)t \quad (5.1)$$

which implies that the phases associated with the propagation of the eigenstates  $\nu_1, \nu_2$  are:

$$\tilde{\phi}_1^\nu = p_1 L - E_1 t, \quad (5.2)$$

$$\tilde{\phi}_2^\nu = p_2 L - E_2 t. \quad (5.3)$$

In the case of pion decay at rest, discussed above, the path length is the same for both mass eigenstates. However, if the times-of-flight are also the same, as assumed in (5.2) and (5.3), then the velocities of the two eigenstates must be the same, which is physically impossible if the neutrinos have different masses. Allowing for different neutrino masses and times-of-flight requires that (5.2) and (5.3) are replaced by:

$$\phi_1^\nu = p_1 L - E_1 t_1 \quad (5.4)$$

$$\phi_2^\nu = p_2 L - E_2 t_2 \quad (5.5)$$

and (5.1) by

$$\Delta\phi_{12}^\nu = (p_1 - p_2)L - E_1 t_1 + E_2 t_2 \quad (5.6)$$

Retaining only the leading  $\mathcal{O}(m_j^2)$  terms in (5.1) gives

$$\begin{aligned} \Delta\tilde{\phi}_{12}^\nu &= (p_1 - p_2)L - (E_1 - E_2)t \\ &= (p_1 - E_1 - p_2 + E_2)L + \mathcal{O}(m_j^4) \\ &= \left( -\frac{m_1^2}{2E_\nu} + \frac{m_2^2}{2E_\nu} \right) L + \mathcal{O}(m_j^4) \\ &= \frac{\Delta m_{21}^2 L}{2E_\nu} + \mathcal{O}(m_j^4) \end{aligned} \quad (5.7)$$

while the same approximation in (5.6) gives [11]:

$$\begin{aligned} \Delta\phi_{12}^\nu &= \left[ p_1 - \frac{E_1}{v_1} - p_2 + \frac{E_2}{v_2} \right] L = \left[ -\frac{m_1^2}{p_1} + \frac{m_2^2}{p_2} \right] L \\ &= \frac{\Delta m_{21}^2 L}{E_\nu} + \mathcal{O}(m_j^4) \end{aligned} \quad (5.8)$$

where the relations  $L = vt$ ,  $v = p/E$  and  $m^2 = E^2 - p^2$  have been used. Writing (5.6) as

$$\begin{aligned}\Delta\phi'_{12} &= (p_1 - p_2)L + (E_1 + E_2)\Delta t - (E_1 - E_2)\bar{t} \\ &= (E_1 - E_2)L + \frac{\Delta m_{21}^2 L}{2E_\nu} + (E_1 + E_2)\Delta t - (E_1 - E_2)L + O(m_j^4) \\ &= \frac{\Delta m_{21}^2 L}{2E_\nu} + (E_1 + E_2)\Delta t + O(m_j^4)\end{aligned}\quad (5.9)$$

where  $\Delta t \equiv (t_2 - t_1)/2$ ,  $\bar{t} \equiv (t_2 + t_1)/2$ , and comparing with (5.8) shows that the  $\Delta t$ -dependent term in (5.9), that is neglected in (5.1), gives a contribution equal to that of the first term. This is the explanation of the factor two difference between the neutrino propagator phase difference (5.8), correctly found by Gribov and Pontecorvo and the standard phase difference of (5.7). Omitting the  $\Delta t$ -dependent term of (5.9) has, fortuitously, the same effect as including the contribution of the pion decay amplitude of Eq. (24) in the Feynman path integral calculation.

## 6 Flavour and Mass Neutrino Eigenstates in Pion Decay

The reason that the same flight time is assigned to both mass eigenstates in the calculation of Ref. [19], as discussed in the previous section, is the hypothesis that what is actually created in the pion decay process is a putative ‘neutrino flavour eigenstate’ with wavefunction  $\psi_{\nu_\mu}$  that is a linear superposition of the wavefunctions of the mass eigenstates:

$$\psi_{\nu_\mu} \equiv U_{\mu 1}\psi_{\nu_1} + U_{\mu 2}\psi_{\nu_2} + U_{\mu 3}\psi_{\nu_3}. \quad (6.1)$$

That is, the invariant amplitudes for the decays  $\pi^+ \rightarrow \bar{\ell}\nu_\ell$ ,  $\bar{\ell} = \mu^+, e^+$  are written, as in the last member of Eq. (2.1), as:

$$\mathcal{M}_{\bar{\ell}} = \frac{G}{\sqrt{2}} f_\pi m_\pi V_{ud} \bar{\psi}_{\bar{\ell}} (1 - \gamma_5) \psi_{\nu_\ell}, \quad \ell = \mu, e \quad (6.2)$$

where  $V_{ij}$  is the Cabibbo-Kobayashi-Maskawa (CKM) [20] quark-flavour mixing matrix and  $G$  is the Fermi constant. On the assumption that all neutrino masses are much smaller than the pion mass, the amplitude in (6.2) may be written in terms of the corresponding ‘reduced amplitude’  $\mathcal{M}_{\bar{\ell}}^0$  for a massless neutrino  $\nu_0$ :

$$\mathcal{M}_{\bar{\ell}} = \mathcal{M}_{\bar{\ell}}^0 [U_{\ell 1} + U_{\ell 2} + U_{\ell 3}], \quad \ell = \mu, e \quad (6.3)$$

where

$$\mathcal{M}_{\bar{\ell}}^0 = \frac{G}{\sqrt{2}} f_\pi m_\pi V_{ud} \bar{\psi}_{\bar{\ell}} (1 - \gamma_5) \psi_{\nu_0}. \quad (6.4)$$

It then follows that [21]:

$$R_{e/\mu} \equiv \frac{\Gamma(\pi^+ \rightarrow e^+ \nu_e)}{\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left[\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right]^2 \left(\frac{U_{e1} + U_{e2} + U_{e3}}{U_{\mu 1} + U_{\mu 2} + U_{\mu 3}}\right)^2. \quad (6.5)$$

Allowing for radiative corrections [22,23] the world average experimental value  $R_{e/\mu} = (1.230 \pm 0.004) \times 10^{-4}$  [8] leads to a constraint on the elements of the PMNS matrix:

$$\left(\frac{U_{e1} + U_{e2} + U_{e3}}{U_{\mu 1} + U_{\mu 2} + U_{\mu 3}}\right)^2 = 0.9976 \pm 0.0032 \quad (6.6)$$

Inserting the values of the PMNS matrix elements from Table 1, taking into account correlations of experimental uncertainties, gives the value  $4.71 \pm 0.32$  for the LHS of Eq. (6.6). It is clear, from these considerations, that the hypothesis that a coherent ‘lepton flavour eigenstate’ is produced in pion decay is experimentally excluded, at the  $11.6\sigma$  level, by the experimental measurements of  $R_{e/\mu}$  and the PMNS matrix elements.

Giunti has claimed [24] that the argument just presented is flawed and that coherent ‘flavour eigenstates’ of massive neutrinos are indeed produced in weak decay processes. To substantiate this claim it is suggested to define a ‘lepton flavour eigenstate’, not according to Eq. (6.1) above, but by instead writing the pion decay amplitude as:

$$\mathcal{M}_{\bar{\ell}}^G = \mathcal{M}_{\bar{\ell}\nu_1} U_{\ell 1} + \mathcal{M}_{\bar{\ell}\nu_2} U_{\ell 2} + \mathcal{M}_{\bar{\ell}\nu_3} U_{\ell 3} \quad (6.7)$$

where  $\mathcal{M}_{\bar{\ell}\nu_j}$  is the invariant amplitude to decay into the mass eigenstate  $\nu_j$ :

$$\mathcal{M}_{\bar{\ell}\nu_j} \equiv \langle \nu_j \bar{\ell} | \pi^+ \rangle \equiv \frac{G}{\sqrt{2}} f_{\pi} m_{\pi} V_{ud} \bar{\psi}_{\bar{\ell}} (1 - \gamma_5) U_{\ell j} \psi_{\nu_j} \simeq \mathcal{M}_{\bar{\ell}}^0 U_{\ell j} \quad j = 1, 2, 3 \quad (6.8)$$

and where in the last member the kinematical effects of non-vanishing neutrino masses have been neglected. Combining (6.7) and (6.8) gives

$$\mathcal{M}_{\bar{\ell}}^G = \mathcal{M}_{\bar{\ell}}^0 [|U_{\ell 1}|^2 + |U_{\ell 2}|^2 + |U_{\ell 3}|^2] = \mathcal{M}_{\bar{\ell}}^0 \quad (6.9)$$

where the unitarity of the PMNS matrix has been invoked. Since the PMNS elements do not appear in Eq. (6.9), the prediction given by this equation for  $R_{e/\mu}$  is the same as the text book massless neutrino result, which is in excellent agreement with experiment and provides no information on the values of the PMNS elements. However, since the amplitude (6.9) has no dependence on the values of these elements, so that, unlike the correct SEM amplitude (6.8), the neutrino mass eigenstates are absent, it does not predict neutrino oscillations following pion decay! This is experimentally excluded by the observation of 2-3 flavour oscillations in both atmospheric neutrinos [25] and the K2K [26] experiment. Actually, the anstaz of Eq. (6.7) which seems to have been constructed precisely to avoid the constraint provided by Eq. (6.6), is in contradiction with the correct SEM expression, (6.8), for the pion decay amplitude, which is linear, not quadratic, in the PMNS elements, and does contain the wavefunction of the mass eigenstate  $\nu_j$  — a necessary consequence of the structure (2.1) of the leptonic charged current in the SEM.

The correct calculation of the pion decay rate assumes independent production of the *physically distinct* mass eigenstates  $\nu_1$  and  $\nu_2$ . Fundamentally, this is because the pion decay process reflects different decay branching ratios of a (virtual) W-boson:  $W \rightarrow \bar{\ell}\nu_1$ ,  $W \rightarrow \bar{\ell}\nu_2$ , which may be compared, for example, to those in the quark sector, described by the CKM matrix  $V_{ij}$ :  $W \rightarrow u\bar{d}$ ,  $W \rightarrow u\bar{s}$ , corresponding to distinct ‘Cabbibo allowed’ and ‘Cabbibo suppressed’ transitions respectively. An analogue, in the quark sector, of the ‘lepton flavour neutrino eigenstate’ of (6.1) would be:

$$\psi_{d_c} = V_{cd}\psi_d + V_{cs}\psi_s + V_{cb}\psi_b \quad (6.10)$$

which is a ‘charm flavour eigenstate of d-type quarks’ comparable to the ‘muon flavour eigenstate of neutrinos’ (6.1). The latter state has no more relevance for leptonic W-boson decays than (6.10) has to hadronic ones.

In the calculation of the pion decay width, the contributions of the different mass eigenstates given by the SEM amplitudes of Eq. (6.8) must be added *incoherently*:

$$\begin{aligned} \Gamma(\pi^+ \rightarrow \mu^+ \nu) &\propto |\mathcal{M}_{\bar{\mu}\nu_1}|^2 + |\mathcal{M}_{\bar{\mu}\nu_2}|^2 + |\mathcal{M}_{\bar{\mu}\nu_3}|^2 \\ &\simeq |\mathcal{M}_{\bar{\mu}}^0|^2 (|U_{\mu 1}|^2 + |U_{\mu 2}|^2 + |U_{\mu 3}|^2) = |\mathcal{M}_{\bar{\mu}}^0|^2. \end{aligned} \quad (6.11)$$

This is in accordance with the quantum mechanical superposition principle [1,2]. Since, unlike in the case of the observed final state (a charged lepton) in neutrino oscillation experiments, the neutrino mass eigenstates are distinct, the contributions of the corresponding decay amplitudes do not interfere. All dependence on the values of the PMNS element vanishes in (6.11) due to the unitarity constraint. Clearly, since decays into the different neutrino mass eigenstates are physically independent processes there is no reason to assume, as in Eq. (5.1), that the decays occur at the same time in the interfering path amplitudes. Indeed, it is essential, if the laws of space time geometry (i.e. the relation  $L = vt_f$ ) are to be respected, that they occur at *different times* in these amplitudes when the ‘neutrino oscillation’ phenomenon occurs.

Although the incoherent nature of the production of the different neutrino mass eigenstates, as exemplified in Eq. (6.11) above, was pointed out more than thirty years ago by Shrock [27,28], and the

unphysical nature of coherent states of neutrinos of different mass was also discussed in the literature [29] the production of a coherent ‘lepton flavour eigenstate’ at a fixed time remains the basic assumption, in the literature, for the calculation of the phase of neutrino oscillations [19]. The assumption that all mass eigenstates are produced at the same time implicitly assumes equal velocities, since there is evidently a unique detection event at some well defined point in space-time. Still, in the derivation of the phase, the neutrino velocities, as defined by the kinematical relation:  $v = p/E$  are assumed to be different. Thus contradictory hypotheses are made in space-time and in momentum space.

Examination of Eq. (4.18) shows that the mechanism that governs the value of  $P_{e\mu\pi}$  is interference between the path amplitudes for different neutrino flavours. A small value of  $P_{e\mu\pi}$  is not necessarily an indication of an approximate conservation of lepton flavour, but may be due to strong destructive interference between the different path amplitudes, independently of the values of the PMNS matrix elements.

The term  $-\cos \Delta\phi_{12}$  in Eq. (4.18) originates in the interference of the path amplitudes corresponding to  $\nu_1$  and  $\nu_2$ . For small values of  $L$ ,  $e^-$  production is suppressed by the almost complete destructive interference of these amplitudes, independently of the value of  $\theta_{12}$  i.e. of the degree of non-conservation of lepton number. The destructive nature of the interference is due to the minus sign multiplying  $\sin\theta_{12}$  in the second row of the matrix on the RHS of Eq. (4.17). This, in turn, is a consequence of the unitarity of the PMNS matrix.

Indeed, nowhere in the description of the ‘ $\nu_e$  appearance’ experiment, described by Eq. (4.18) do ‘lepton flavour eigenstates’ occur, although such an experiment is typically referred to [19] as ‘ $\nu_\mu \rightarrow \nu_e$  flavour oscillation’. In fact, only the mass eigenstates  $\nu_1, \nu_2$  appear in the amplitudes of the physical processes which interfere. It is the interference of these amplitudes in the production of the detection event that constitutes the phenomenon of ‘neutrino oscillations’; no temporal oscillations of ‘lepton flavour’ actually occur. Within each path amplitude the neutrino is in a definite mass eigenstate. The so-called ‘oscillation’ phenomenon is an attribute of the detection process where interference occurs between the different path amplitudes, each corresponding to a definite neutrino mass eigenstate, in the production of a charged lepton of definite flavour. Still the terms ‘ $\nu_e$ ’, ‘ $\nu_\mu$ ’ and ‘ $\nu_\tau$ ’ may still have a certain utility as mnemonics, even though they do not represent physical neutrino states for massive neutrinos. For example, it makes sense to refer to solar neutrinos, in a loose way, as a ‘ $\nu_e$  beam’ since the different physical components are all created together with an electron. Similarly, atmospheric neutrinos are predominantly ‘ $\nu_\mu$ ’, i.e., born together with a muon.

## 7 Probability Amplitude Analysis of Other Related Experiments

The different ingredients —the amplitudes (i)-(iv) above— that contribute to the path amplitudes in Feynman’s formulation of quantum mechanics, have all been experimentally verified in various two-path quantum interference experiments apart from neutrino oscillations. There is no reason to suppose that the laws of physics governing the latter should be any different than in neutrino oscillations <sup>4</sup>.

The existence of the contribution (ii) —the decay amplitude of the unstable source particle— with time intervals  $t_f$  calculated according to exact space time geometry:  $t_f = s/v$  where  $s$  is the path length and  $v$  the free-particle velocity, is verified by:

- All diffraction and interference experiments in photonic optics [4,2,3]. In this case the entire interference phase originates in the decay amplitude, (ii), of the source, since, as shown above, the space-time propagator of the photon does not change the phase of the path amplitude.

<sup>4</sup> This is an application of the second of Newton’s ‘Rules for the study of natural philosophy’: ‘...the causes assigned to natural effects of the same kind must be, as far as possible, the same.’ [I. Newton, *The Principia*, Translated by I.B. Cohen and A. Whitman, (University of California Press, Berkeley 1999), p.795.]. In the present case it seems reasonable that ‘the causes’ of all two path quantum mechanical experiments should be the same. Then all such experiments either should, or should not, be correctly described by path integrals. It seems that they are so described. On the contrary, if the equal-time ansatz of Eq. (5.1), that gives the same prediction as the Feynman path approach for neutrino oscillations, is applied to photonic physical optics it predicts the vanishing of all interference and diffraction effects, at variance with experiment and in contradiction to the Rule.

- The quantum beat experiment [32,12]. This experiment measures directly the phase variation of the decay amplitude given by Eq. (3.1) for excited atoms. A beam of atoms is excited into different states by interaction with a thin foil (Coulomb excitation) or a laser beam. A decay photon detected downstream may originate from different excited states. Interference of the corresponding path amplitudes gives a cosine term in the photon detection rate as a function of the distance  $d$  from the excitation foil with a phase:

$$\phi_{\text{beat}} = \frac{(E_{\alpha}^* - E_{\beta}^*)d}{\bar{v}_{\text{atom}}}$$

where  $E_{\alpha}^*$  and  $E_{\beta}^*$  are the energies of two excited states and  $\bar{v}_{\text{atom}}$  is the average velocity of the atomic beam. This experiment is a direct test of the correctness of Eq. (3.1).

The contribution of the propagator of a massive particle, (iii), is demonstrated by:

- The Young double slit experiment using electrons. In this case there is no coherent electron source. The detailed space-time analysis [2] shows that the interference effect requires finite-width momentum wave packets, the observed interference wavelength corresponding to equal production times and different velocities in the two interfering path amplitudes. The interference phase thus originates entirely from the electron propagator, in contrast to the double slit experiment with photons, where only the source particle decay amplitude contributes. In both cases the Feynman path integral analysis predicts purely spatial classical wave theories with well defined momentum-dependent wavelengths, in the case that the lifetime of any coherent source is much greater than the difference between the times-of-flight in the two paths [2,3].

The combined effect, in the same experiment, of the amplitudes (ii) and (iii) is demonstrated by:

- The photodetachment microscope [12,33,34,35]. Here a coherent source of electrons of fixed energy is provided by a negative ion beam irradiated by a laser. The detached electron moves in a constant external electric field before detection. Just two classical trajectories link the point of emission to any point on a plane detector oriented perpendicularly to the electric field direction. Quantum interference effects are observed between the path amplitudes corresponding to the two trajectories. A good pedagogical description can be found in Ref. [35] where the appropriate path integral formula <sup>5</sup>:

$$\psi(\mathbf{r}, t_f) = \int_{-\infty}^{t_f} \exp[-i\frac{\epsilon t_i}{\hbar}] \exp[i\frac{S_{cl}(\mathbf{r}, t_i, t_f)}{\hbar}] dt_i$$

is given.

In this formula  $\epsilon$  is the energy of the detached electron and  $S_{cl}$  the classical action corresponding to an electron trajectory. Note particularly the time integral on the RHS of the equation. The first exponential function is the amplitude of the coherent source, as given by Eq. (3.1); the second represents the propagator of the electron in the electric field. In practice it is well approximated by the contributions of the two classical trajectories mentioned above, corresponding to values of  $t_i$  with a fixed separation. These are the analogues of the propagators of different neutrino mass eigenstates. A typical value of the difference in  $t_i$  between the two trajectories, quoted in Ref. [35] is 160 psec for a time-of-flight of 117 nsec.

The laws of physics must be the same in all of the above ‘two path’ quantum mechanical experiments and in any neutrino oscillation experiment. In particular, the contribution of the source amplitude (ii) is essential for the derivation of the standard oscillation phase of Eq. (4.20) that has hitherto been obtained in a manner that does not respect Feynman’s formulation of the laws of quantum mechanics [1,2], but that, fortuitously, obtains the same result as the calculation, presented above, that does.

<sup>5</sup> A similar formula was proposed for the neutrino oscillation problem in V.Pařma and J.Vanko, Alternative Theory of Neutrino Oscillations, <http://arxiv.org/abs/hep-ph/0311090>. The corresponding oscillation phase was not, however, derived.

## 8 Neutrino Oscillations and Quantum Field Theory

Neutrino oscillations have been described in the literature [36] using the formalism of quantum field theory (QFT) in which quantum field operators are written as a superposition of momentum-dependent creation and annihilation operators for specific particles [37]. The quantum wavefunctions of these particles are then obtained from the vacuum state of the theory by application to it of the quantum field creation operators, while application of the appropriate annihilation operator to a particle wavefunction recovers the vacuum state. In order to relate such QFT-approaches to that of the present paper a few general remarks are made concerning the relation of the Feynman path integral to QFT, as well as how predictions for specific physical processes such as those discussed above, are obtained from QFT.

The essential physics of any QFT such as QED or the SEM is contained in the Lagrangian of the model written in terms of the quantum fields of the particles of the theory. As shown, for example, in Ramond's textbook on QFT [39] or by Weinberg [40] the Feynman rules of a QFT may be derived from the Feynman path integral containing its field Lagrangian. Given the Feynman rules, the invariant amplitude for any process involving interactions between the particles of the theory can be calculated; for example the pion decay amplitude of Eq. (6.8). In the calculation of the invariant amplitude according to the Feynman rules, only wavefunctions and propagators of particles are considered—not quantum field operators. This result—that all testable physical predictions of a QFT are implicit in its Feynman rules—was, of course, the seminal contribution of Feynman to the subject. In the present context this means that predictions for neutrino oscillations can be derived, in complete generality, from the Feynman rules of the SEM without the necessity for any consideration of QFT concepts such as 'second quantised' field operators<sup>6</sup>. The Feynman path integral is therefore not only the basis of a new space-time formulation of the principles of quantum mechanics [1,2] but also, when applied to QFT, an elegant route to the derivation of the Feynman rules—and hence all physical predictions—of the theory. As pointed out by Dirac [41] and reiterated by Feynman [42] the quantum mechanical path integral also predicts, in the limit where Planck's constant,  $h$ , is very small as compared to the classical action,  $S$ , of the problem considered, Hamilton's principle, which constitutes a complete dynamical basis for classical mechanics.

## 9 Conclusions

According to the Feynman rules of the SEM only neutrino mass eigenstates are created, destroyed, or propagate in space-time. In general the mass eigenstates are produced at different times in different decay amplitudes, and the constraints of relativistic kinematics and classical space-time geometry are rigorously respected. In accordance with the superposition law of quantum mechanics [1,2] the neutrino mass eigenstates are physically distinct (not components of a 'neutrino flavour wavefunction') and therefore produced *incoherently*, in any decay process in which they are produced, as in Eq. (6.11) above. This was clearly stated by Shrock, some three decades ago now [27]:

" Thus, for example, a decay of the form  $X \rightarrow Y = \ell_a + \bar{\nu}_{\ell_a}$  would actually consist of an incoherent sum of the separate modes  $X \rightarrow Y = \ell_a + \bar{\nu}_i$ , where  $i$  runs over the subset of neutrino mass eigenstates allowed by phase space."

The comparison of neutrino oscillations with the Young double slit interference experiment presented here is important because:

- (i) It made evident an inconsistent postulate in previous work by the present author that found a non-standard result for the neutrino oscillation phase in Refs. [11,12,2] and in several related arXiv preprints [13,14].
- (ii) It showed that the postulates of the 'plane wave' calculation of neutrino oscillations are incompatible with optical interference effects, whereas the Feynman path amplitude analysis gives a consistent description of neutrino oscillations, physical optics and many other problems where quantum interference effects are important, some of which are described in Section 7 above, or in Ref. [2].

<sup>6</sup> It is interesting to note that in Feynman's first QED paper: R.P.Feynman, Phys. Rev. **76** (1949) 749, where the 'Feynman diagram' concept was introduced, an appendix was added where the principal results were rederived using conventional second quantised QFT, invoking creation and annihilation operators. In this way the referee of the paper (presumably an expert in the conventional QFT of that time) could be assured of the correctness of results obtained in a more elegant and rapid way by the Feynman diagram technique.

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