Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters

Pierre Gaillard

Université de Bourgogne, Institut de mathématiques de Bourgogne, 9 avenue Alain Savary BP 47870 - 21078 Dijon Cedex, France
Email: Pierre.Gaillard@u-bourgogne.fr

Abstract We give different representations of the solutions of the Johnson equation with parameters. First, an expression in terms of Fredholm determinants is given; we give also a representation of the solutions written as a quotient of wronskians of order $2N$. These solutions of order N depend on $2N-1$ parameters. When one of these parameters tends to zero, we obtain N order rational solutions expressed as a quotient of two polynomials of degree $2N(N+1)$ in x, t and $4N(N+1)$ in y depending on $2N-2$ parameters.
Here, we explicitly construct the expressions of the rational solutions of order 5 depending on 8 real parameters and we study the patterns of their modulus in the plane (x,y) and their evolution according to time and parameters a_i and b_i for $1 \leq i \leq 4$.

Keywords: Johnson equation, Fredholm determinants, wronskians, rational solutions, rogue waves.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the Johnson equation which can be written in the form

$$(u_t + 6uu_x + u_{xxx} + \frac{u}{2t})_x - 3\frac{u_{yy}}{t^2} = 0, \tag{1}$$

where subscripts x, y and t denote partial derivatives.

Johnson introduced this equation in a paper written in 1980 [1] to describe waves surfaces in shallow incompressible fluids [2,3]. This equation was widely accepted, and was later derived for internal waves in a stratified medium [4]. The physical model of this equation have the same degree of universality as the Kadomtsev-Petviashvili (KP) equation [5].

Johnson constructed the first solutions in 1980 [1]. Some time later in 1984, GolinŠko, Dryuma, and Stepanyants found other types of solutions [6]. Another approach to study this equation was given in 1986 [7] by giving a connection between solutions of the (KP) equation and solutions of the Johnson equation. The use of Darboux transformation gave other type of solutions given in [8]. More recently, the extension to the elliptic case has been considered [9] in 2013.

In the following, we recall the representation of the solutions in terms of Fredholm determinants of order $2N$ depending on $2N-1$ parameters. We also recall the expression in terms of wronskians of order $2N$ with $2N-1$ parameters. These representations allow to obtain an infinite hierarchy of solutions to the Johnson equation, depending on $2N-1$ real parameters and rational solutions to the equation, when a parameter tends towards 0.

Here we construct rational solutions of order 5 depending on 8 parameters, and the representations of their modulus in the plane of the coordinates (x,y) according to the real parameters a_i and b_i for $1 \leq i \leq 4$ and time t.

The solutions are given without initial conditions nor boundary conditions.

We give three methods to construct solutions to the Johnson equation. The more efficient method to construct solutions of the Johnson equation is that corresponding to the representation in terms of degenerate determinants (the third one in the text, without limit) followed by that given in terms of wronskians. The less efficient is that given in terms of Fredholm determinants.

The method used to construct the figures given in the third section is that using the degenerate determinants (without limit, the third one).

2 Rational Solutions to the Johnson Equation of Order N Depending on $2N-2$ Parameters

2.1 Families of Rational Solutions of Order N Depending on $2N-2$ Parameters

We define real numbers λ_j such that $-1<\lambda_\nu<1$, $\nu=1,\ldots,2N$ which depend on a parameter ϵ which will be intended to tend towards 0; they can be written as

$$\lambda_j = 1-2\epsilon^2 j^2, \quad \lambda_{N+j}=-\lambda_j, \quad 1\leq j\leq N, \tag{2}$$

The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of λ_ν, $1\leq\nu\leq 2N$; they are defined by the formulas:

$$\begin{array}{l}
\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j=\kappa_j\lambda_j, \quad \gamma_j=\sqrt{\frac{1-\lambda_j}{1+\lambda_j}},; \\
x_{r,j}=(r-1)\ln\frac{\gamma_j-i}{\gamma_j+i}, \quad r=1,3, \quad \tau_j=-12i\lambda_j^2\sqrt{1-\lambda_j^2}-4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2}, \\
\kappa_{N+j}=\kappa_j, \quad \delta_{N+j}=-\delta_j, \quad \gamma_{N+j}=\gamma_j^{-1}, \\
x_{r,N+j}=-x_{r,j},, \quad \tau_{N+j}=\tau_j \quad j=1,\ldots,N.
\end{array} \tag{3}$$

e_ν $1\leq\nu\leq 2N$ are defined in the following way:

$$\begin{array}{l}
e_j = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} - i\sum_{k=1}^{1/2\,M-1} b_K (je)^{2\,k+1}\right), \\
e_{N+j} = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} + i\sum_{k=1}^{1/2\,M-1} b_k (je)^{2\,k+1}\right), \quad 1\leq j\leq N, \\
a_k, b_k \in \mathbf{R}, \quad 1\leq k\leq N.
\end{array} \tag{4}$$

ϵ_ν, $1\leq\nu\leq 2N$ are real numbers defined by:

$$\epsilon_j=1, \quad \epsilon_{N+j}=0 \quad 1\leq j\leq N. \tag{5}$$

Let I be the unit matrix and $D_r=(d_{jk})_{1\leq j,k\leq 2N}$ the matrix defined by:

$$d_{\nu\mu} = (-1)^{\epsilon_\nu}\prod_{\eta\neq\mu}\left(\frac{\gamma_\eta+\gamma_\nu}{\gamma_\eta-\gamma_\mu}\right)\exp(\kappa_\nu x+(\frac{\kappa_\nu y}{12}-2\delta_\nu)yt+4i\tau_\nu t+x_{r,\nu}+e_\nu). \tag{6}$$

Then we have the following result:

Theorem 2.1 *The function v defined by*

$$v(x,y,t) = -2\,\frac{|n(x,y,t)|^2}{d(x,y,t)^2} \tag{7}$$

where

$$n(x,y,t)=\det(I+D_3(x,y,t)), \tag{8}$$

$$d(x,y,t)=\det(I+D_1(x,y,t)), \tag{9}$$

and $D_r=(d_{jk})_{1\leq j,k\leq 2N}$ the matrix

$$d_{\nu\mu} = (-1)^{\epsilon_\nu}\prod_{\eta\neq\mu}\left(\frac{\gamma_\eta+\gamma_\nu}{\gamma_\eta-\gamma_\mu}\right)\exp(\kappa_\nu x+(\frac{\kappa_\nu y}{12}-2\delta_\nu)yt+4i\tau_\nu t+x_{r,\nu}+e_\nu). \tag{10}$$

is a solution to the Johnson equation (1), depending on $2N-1$ parameters a_k, b_h, $1\leq k\leq N-1$ and ϵ.

We give now the expressions of the solutions to the Johnson equation in terms of wronskians. For this, we define the following notations:

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1,3, \tag{11}$$

with the arguments

$$\Theta_{r,\nu} = \tfrac{-i\kappa_\nu x}{2} + i(\tfrac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\tfrac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\tfrac{e_\nu}{2}, \quad 1 \leq \nu \leq 2N. \tag{12}$$

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu\in[1,\dots,2N]}]. \tag{13}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\dots,2N]}$ defined in (10).

Then we have the following statement:

Theorem 2.2 *The function v defined by*

$$v(x,y,t) = -2\frac{|W_3(\phi_{3,1},\dots,\phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1},\dots,\phi_{1,2N})(0))^2}$$

is a solution to the Johnson equation depending on $2N-1$ real parameters a_k, b_k and ϵ, with ϕ^r_ν defined in (11)

$$\begin{aligned}
&\phi_{r,\nu} = \sin(\tfrac{-i\kappa_\nu x}{2} + i(\tfrac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\tfrac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\tfrac{e_\nu}{2}), \quad 1 \leq \nu \leq N,\\
&\phi_{r,\nu} = \cos(\tfrac{-i\kappa_\nu x}{2} + i(\tfrac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\tfrac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\tfrac{e_\nu}{2}), \quad N+1 \leq \nu \leq 2N, \quad r = 1,3,
\end{aligned}$$

κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in (3), (2) and (4).

We can deduce rational solutions to the Johnson equation as a quotient of two determinants.

We use the following notations:

$$X_\nu = \frac{-i\kappa_\nu x}{2} + i(\frac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\frac{x_{3,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2},$$

$$Y_\nu = \frac{-i\kappa_\nu x}{2} + i(\frac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\frac{x_{1,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2},$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (3) and parameters e_ν defined by (4). We define the following functions:

$$\begin{aligned}
\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k,\\
\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,
\end{aligned} \tag{14}$$

for $1 \leq k \leq N$, and

$$\begin{aligned}
\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k},\\
\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k},
\end{aligned} \tag{15}$$

for $1 \leq k \leq N$. We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k.

$$\begin{aligned}
\psi_{4j+1,k} = \gamma_k^{4j-1}\sin Y_k, \quad \psi_{4j+2,k} = \gamma_k^{4j}\cos Y_k,\\
\psi_{4j+3,k} = -\gamma_k^{4j+1}\sin Y_k, \quad \psi_{4j+4,k} = -\gamma_k^{4j+2}\cos Y_k,
\end{aligned} \tag{16}$$

for $1 \leq k \leq N$, and

$$\begin{aligned}
\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin Y_{N+k},\\
\psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin Y_{N+k},
\end{aligned} \tag{17}$$

for $1 \leq k \leq N$.

The following ratio

$$q(x,t) := \frac{W_3(0)}{W_1(0)}$$

can be written as

$$q(x,t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,\,k\in[1,2N]}}{\det(\psi_{j,k})_{j,\,k\in[1,2N]}}. \tag{18}$$

The terms λ_j depending on ϵ are defined by $\lambda_j = 1 - 2j\epsilon^2$. All the functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ. They can all be prolonged by continuity when $\epsilon = 0$.

We use the following expansions

$$\varphi_{j,k}(x,y,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\varphi_{j,1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,1}[l] = \frac{\partial^{2l}\varphi_{j,1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\varphi_{j,1}[0] = \varphi_{j,1}(x,y,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\varphi_{j,N+k}(x,y,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\varphi_{j,N+1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,N+1}[l] = \frac{\partial^{2l}\varphi_{j,N+1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\varphi_{j,N+1}[0] = \varphi_{j,N+1}(x,y,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1.$$

We have the same expansions for the functions $\psi_{j,k}$.

$$\psi_{j,k}(x,y,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\psi_{j,1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,1}[l] = \frac{\partial^{2l}\psi_{j,1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\psi_{j,1}[0] = \psi_{j,1}(x,y,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\psi_{j,N+k}(x,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\psi_{j,N+1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,N+1}[l] = \frac{\partial^{2l}\psi_{j,N+1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\psi_{j,N+1}[0] = \psi_{j,N+1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad N+1 \leq k \leq 2N..$$

Then we get the following result:

Theorem 2.3 *The function v defined by*

$$v(x,y,t) = -2\frac{|\det((n_{jk})_{j,k\in[1,2N]})|^2}{\det((d_{jk})_{j,k\in[1,2N]})^2} \tag{19}$$

is a rational solution to the Johnson equation (1), where

$$\begin{array}{l}
n_{j1} = \varphi_{j,1}(x,y,t,0),\, 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
n_{jN+1} = \varphi_{j,N+1}(x,y,t,0),\, 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
d_{j1} = \psi_{j,1}(x,y,t,0),\, 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
d_{jN+1} = \psi_{j,N+1}(x,y,t,0),\, 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
2 \leq k \leq N,\, 1 \leq j \leq 2N
\end{array} \tag{20}$$

The functions φ and ψ are defined in (14),(15), (16), (17).

3 Explicit Expression of Rational Solutions of Order 5 Depending on 8 Parameters

We construct rational solutions to the Johnson equation of order 5 depending on 8 parameters. But, because of the length of the expression, we cannot get them in this text.

We give patterns of the modulus of the solutions in the plane (x, y) of coordinates in function of the parameters a_i and b_i, for $1 \leq i \leq 4$ and time t.

The $(x; y)$ plane is the horizontal plane. To shorten the text, one cut certain characters of the figures and one made appear only the letter y of the $(x; y)$ plane.

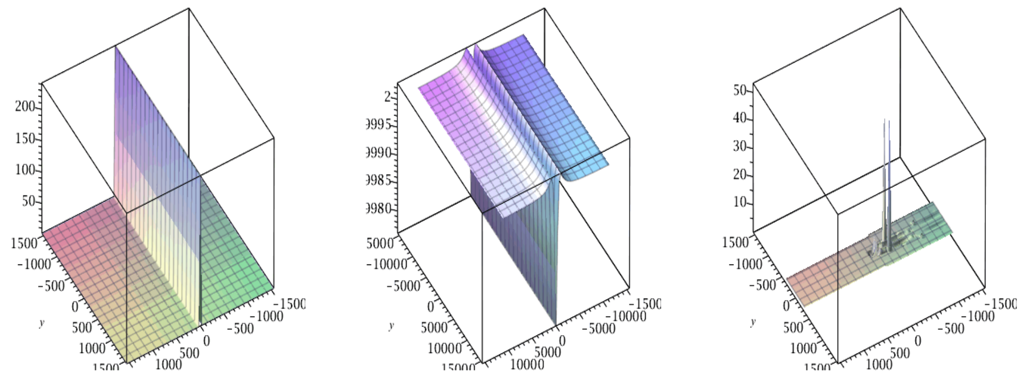

Figure 1. Solution of order 5 to (1), on the left for $t = 0$; in the center for $t = 0$, $a_1 = 10^3$; on the right for $t = 0$, $a_2 = 10^3$; all other parameters not mentioned equal to 0.

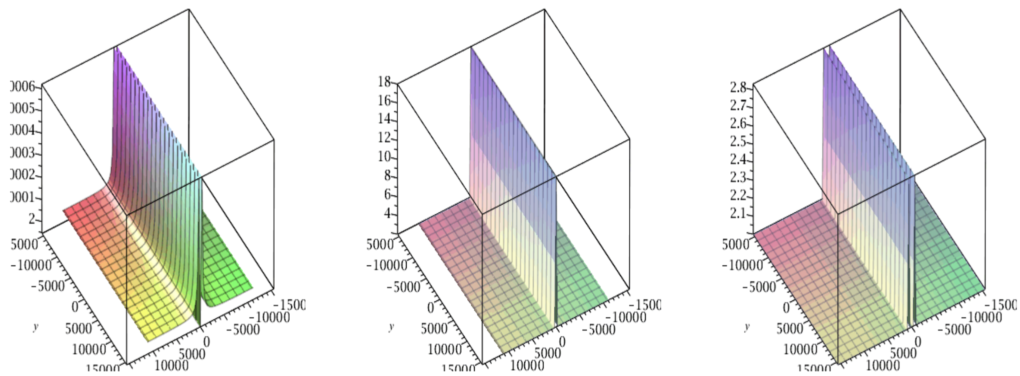

Figure 2. Solution of order 5 to (1), on the left for $t = 0$, $a_3 = 10^3$; in the center for $t = 0$, $a_4 = 10^3$; on the right for $t = 0$, $b_1 = 10^3$; all other parameters not mentioned equal to 0.

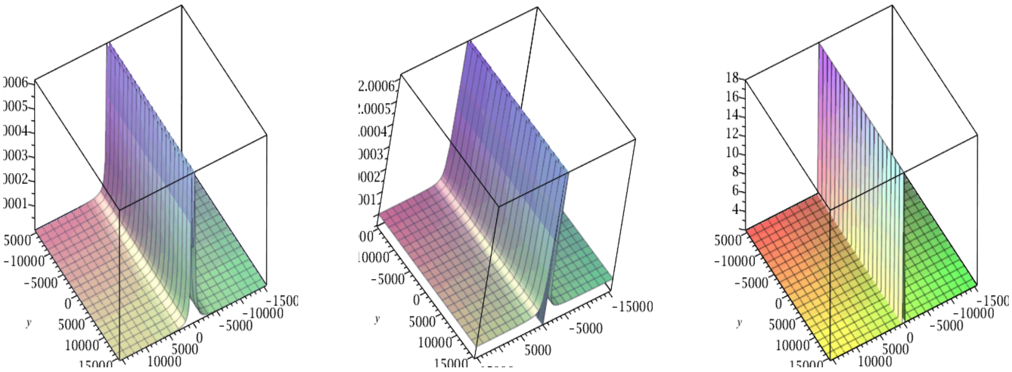

Figure 3. Solution of order 5 to (1), on the left for $t = 0$, $b_2 = 10^3$; in the center for $t = 0$, $b_3 = 10^3$; on the right for $t = 0$, $b_4 = 10^3$; all other parameters not mentioned equal to 0.

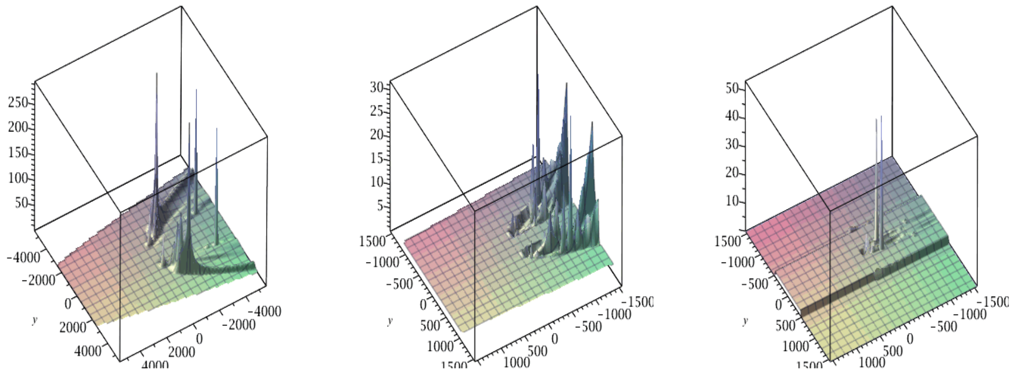

Figure 4. Solution of order 5 to (1), on the left for $t = 0,01$, $a_1 = 10^3$; in the center for $t = 0,1$, $a_2 = 10^3$; on the right for $t = 1$, $b_1 = 10^3$; all other parameters not mentioned equal to 0.

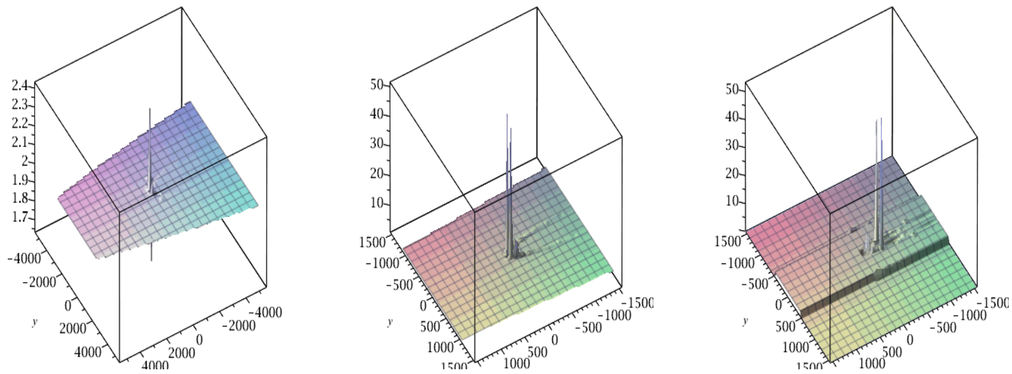

Figure 5. Solution of order 5 to (1), on the left for $t = 0,01$, $a_2 = 10^3$; in the center for $t = 0,1$, $a_2 = 10^3$; on the right for $t = 1$, $a_2 = 10$; all the other parameters to equal to 0.

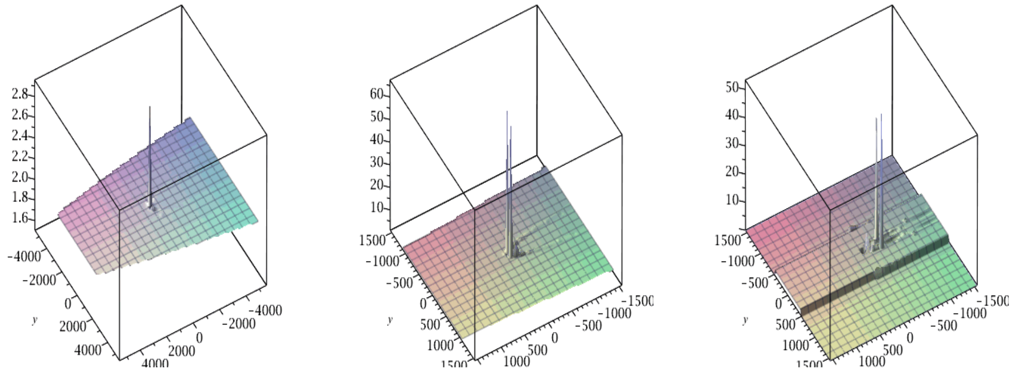

Figure 6. Solution of order 5 to (1), on the left for $t = 0,01$, $a_3 = 10^3$; in the center for $t = 0,1$, $a_3 = 10^3$; on the right for $t = 1$, $a_3 = 10$; all the other parameters to equal to 0.

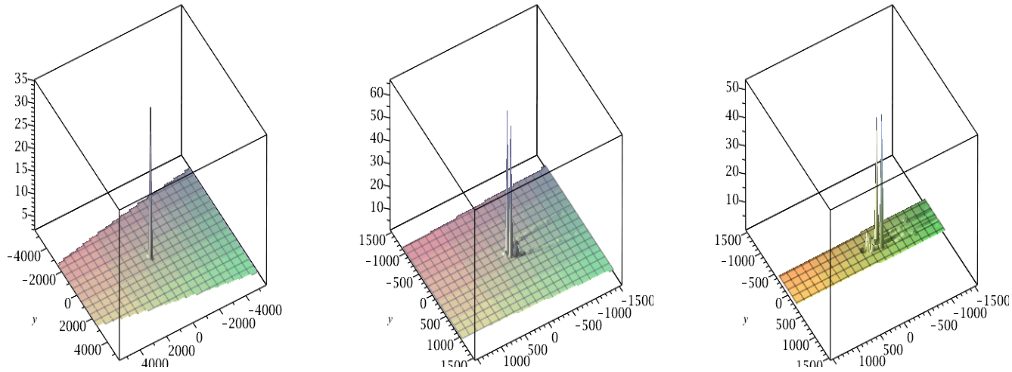

Figure 7. Solution of order 5 to (1), on the left for $t = 0,01$, $a_4 = 10^3$; in the center for $t = 0,1$, $a_4 = 10^3$; on the right for $t = 1$, $a_4 = 10$; all the other parameters to equal to 0.

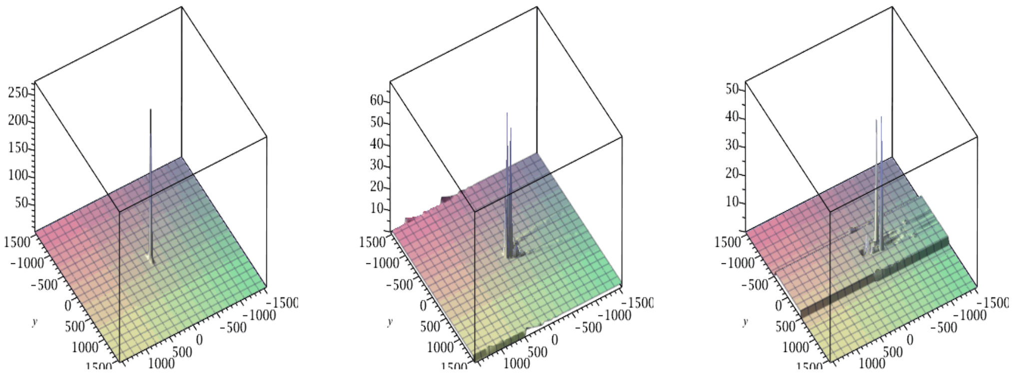

Figure 8. Solution of order 5 to (1), on the left for $t = 0,01$, $b_1 = 10$; in the center for $t = 0,1$, $b_4 = 10$; on the right for $t = 1$, $b_1 = 10$; all the other parameters to equal to 0.

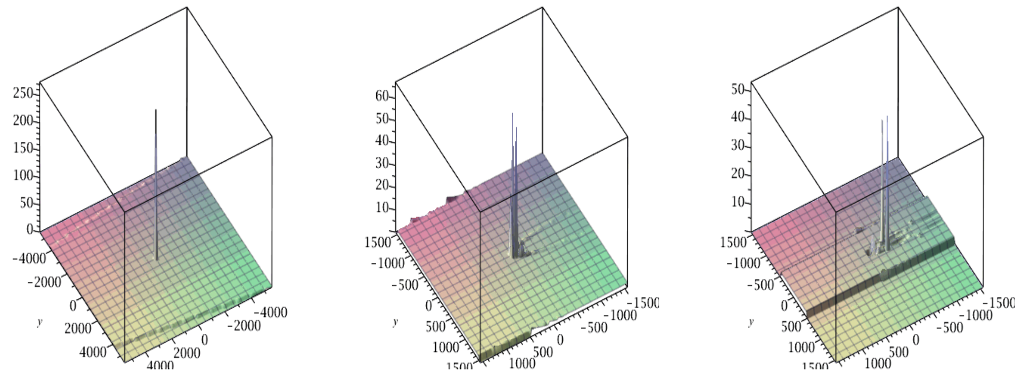

Figure 9. Solution of order 5 to (1), on the left for $t = 0,01$, $b_2 = 10^3$; in the center for $t = 0,1$, $b_2 = 10$; on the right for $t = 1$, $b_2 = 10$; all the other parameters to equal to 0.

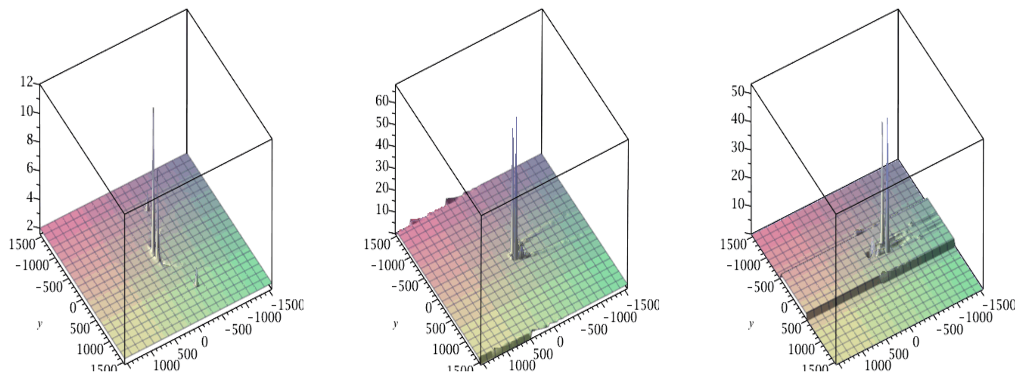

Figure 10. Solution of order 5 to (1), on the left for $t = 0,01$, $b_3 = 10^3$; in the center for $t = 0,1$, $b_3 = 10^3$; on the right for $t = 1$, $b_3 = 10^3$; all the other parameters to equal to 0.

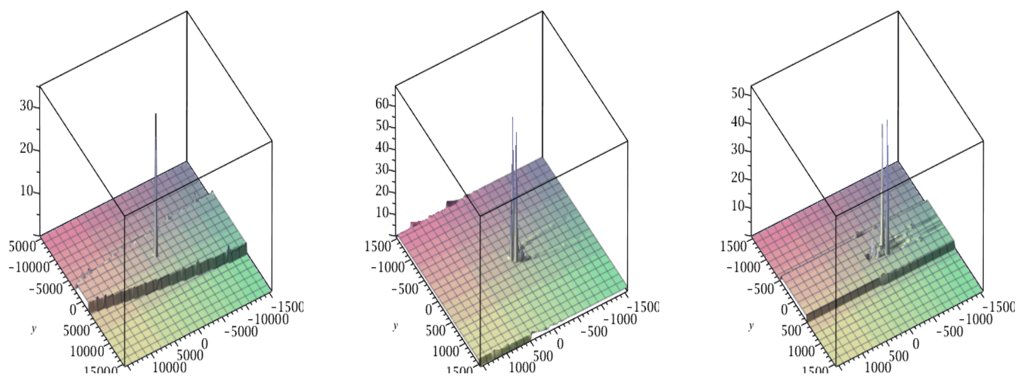

Figure 11. Solution of order 5 to (1), on the left for $t = 0,01$, $b_4 = 10^3$; in the center for $t = 0,1$, $b_4 = 10^3$; on the right for $t = 1$, $b_4 = 10$; all the other parameters to equal to 0.

In these constructions, we note that the initial rectilinear structure becomes deformed very quickly as time t increases. The heights of the peaks also decrease very quickly according to time t and of the various parameters. Because of the structure of the polynomials, one notices that the modulus of these solutions tend towards value 2 when time t and variables x and y tend towards the infinite.

The preceding solutions depend on parameters a_j and b_j for $1 \leq j \leq 4$. The Johnson equation allows explaining the existence of the horseshoelike solitons and multisoliton solutions quite naturally. The horseshoe multisoliton solutions correspond very well to real waves observed in thin films of shallow water being cooled along an inclined plane.

It should be relevant to give a physical meaning of these parameters and to give an explanation of the evolution of the figures according to time in the $(x; y)$ plane.

4 Conclusion

We succeed in obtaining rational solutions to the Johnson equation depending on $2N-2$ real parameters. These solutions can be expressed in terms of a ratio of two polynomials of degree $2N(N+1)$ in x, t and $4N(N+1)$ in y. Here we have made the study of rational solutions of order 5 depending on 8 parameters and tried to describe the structure of those rational solutions.

In the $(x; y)$ plane of coordinates, various structures appear. But, contrary to the rational solutions of the NLS or KP equations, there are not well defined structures which appear according to the parameters a_i or b_i. Thus, one cannot carry out a classification of these solutions here, according to the parameters by means of their module in the plan (x, y). It would be important to better understand these structures.

References

1. R.E. Johnson, Water waves and KortewegŰde Vries equations, J. Fluid Mech., V. 97, N. 4, 701Ű719, 1980
2. R.E. Johnson, A Modern Introduction to the Mathematical Theory of Water Waves, Cambridge University Press, Cambridge, 1997
3. M. J. Ablowitz, Nonlinear Dispersive Waves : Asymptotic Analysis and Solitons, Cambridge University Press, Cambridge, 2011
4. V.D. Lipovskii1, On the nonlinear internal wave theory in fluid of finite depth, Izv. Akad. Nauka., V. 21, N. 8, 864Ű871, 1985
5. B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, Sov. Phys. Dokl., V. 15, N. 6, 539-541, 1970
6. V.I. GolinŠko, V.S. Dryuma, Yu.A. Stepanyants, Nonlinear quasicylindrical waves: Exact solutions of the cylindrical Kadomtsev- Petviashvili equation, in Proc. 2nd Int. Workshop on Nonlinear and Turbulent Processes in Physics, Kiev, Harwood Acad., Gordon and Breach, 1353Ű1360, 1984
7. V.D. Lipovskii, V.B. Matveev, A.O. Smirnov, Connection between the Kadomtsev-Petvishvili and Johnson equation, Zap. Nau. Sem., V. 150, 70Ű75, 1986
8. C. Klein, V.B. Matveev, A.O. Smirnov, Cylindrical Kadomtsev-Petviashvili equation: Old and new results, Theor. Math. Phys., V. 152, N. 2, 1132-1145, 2007
9. K. R. Khusnutdinova, C. Klein, V.B. Matveev, A.O. Smirnov, On the integrable elliptic cylindrical K-P equation Chaos, V. 23, 013126-1-15, 2013
10. P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, J. Phys. A : Meth. Theor., V. 44, 1-15, 2010
11. P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, Jour. Of Math. Phys., V. 54, 2013, 013504-1-32
12. P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, Adv. Res., V. 4, 2015, pp-346-364
13. P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, Jour. of Math. Phys., V. 57, 063505-1-13, 2016
14. P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, Theo. And Mat. Phys., V. 196, N. 2, 1174-1199, 2018

